COMPLEX POWERS OF i SATISFYING THE CONTINUED FRACTION FUNCTIONAL EQUATION OVER THE GAUSSIAN INTEGERS

Abstract. We investigate and then state the conditions under which i^z satisfies the simple continued fraction functional equation for real and then complex z over the Gaussian integers.

1. Introduction

Consider the functional equation

$$f(z) = u + \frac{1}{f(z)} \tag{1.1}$$

For some $z_0 \in \mathbb{C}$ we say that $f(z_0)$ satisfies the simple continued fraction equation over $S \subset \mathbb{C}$ if (1.1) is satisfied for $z = z_0$ and some $u_0 \in S$. Note that if (1.1) holds for some $z_0, u_0 \in \mathbb{C}$, this does not imply the convergence of

$$u_0 + \cfrac{1}{u_0 + \cfrac{1}{\ddots}} \tag{1.2}$$

In this short note, we investigate the conditions under which $f(z) = i^z$ has a continued fraction representation over the Gaussian integers for real and then complex z. That is, we find the z for which

$$i^z = u + \frac{1}{i^z} \tag{1.3}$$

With u a Gaussian integer. We find the finite number of solutions to (1.3) in the former case, and a general form for solution z to (1.3) given u in the latter.

2. Real Powers

Notice that

$$i^z = u + \frac{1}{i^z} \implies (-1)^z - 1 = ui^z \tag{2.1}$$

Which, given $|i^x| = 1$ for $x \in \mathbb{R}$, we may apply the triangle inequality to to obtain

$$|(-1)^z| + |-1| \geqslant |u||i^z| \implies 2 \geqslant |u| \tag{2.2}$$

Limiting the possible values of u to thirteen Gaussian integers, assuming z is real. One may check each possible value by casework to produce twenty-some sets of solutions z to (1.3), not all of which are real. We will not do this, instead making use of the identity

$$i^z = \cos(\frac{\pi z}{2}) + i\sin(\frac{\pi z}{2}) \tag{2.3}$$

Which holds for $z \in \mathbb{R}$. Rearranging (1.3), we have

$$i^z - i^{-z} = u \tag{2.4}$$

With $\sin(\cdot)$ and $\cos(\cdot)$ being odd and even functions, respectively, substituting (2.3) into (2.4) gives us

$$2i\sin(\frac{\pi z}{2}) = u \tag{2.5}$$

Then, as z is restricted to real values and both i^z and (2.5) have period 4, we need only consider $z \in \{0, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm 1\}$, the values of z for which $\sin(\frac{\pi z}{2})$ is an integer. Raising i to each of these powers gives us 1 and the sixth roots of -1, the former we exclude hereafter as a trivial case. It follows that the sixth roots of -1 are the non-trivial real powers of i which satisfy (1.3) for u a Gaussian integer.

We now solve for u for each possible value of z. Substituting (2.5) into (1.3), we obtain

$$i^z = 2i\sin(\frac{\pi z}{2}) + \frac{1}{i^z} \tag{2.6}$$

Into which we substitute the well-known equivalence $i^x = e^{\frac{1}{2}\pi i x}$ to obtain

$$e^{\frac{1}{2}\pi i z} = 2i\sin(\frac{\pi z}{2}) + \frac{1}{e^{\frac{1}{2}\pi i z}} \tag{2.7}$$

Which we use for its convenience in distinguishing the sixth roots of -1. Then, evaluating (2.7) at each possible value of z, we have

$$e^{\frac{1}{6}\pi i} = i + \frac{1}{e^{\frac{1}{6}\pi i}} \tag{2.8}$$

$$e^{-\frac{1}{6}\pi i} = -i + \frac{1}{e^{-\frac{1}{6}\pi i}} \tag{2.9}$$

$$e^{\frac{5}{6}\pi i} = i + \frac{1}{e^{\frac{5}{6}\pi i}} \tag{2.10}$$

$$e^{-\frac{5}{6}\pi i} = -i + \frac{1}{e^{-\frac{5}{6}\pi i}} \tag{2.11}$$

$$i = 2i + \frac{1}{i} \tag{2.12}$$

$$-i = -2i + \frac{1}{-i} \tag{2.13}$$

As the six non-trivial[1] real powers of i satisfying (1.3) for $u \in \mathbb{Z}[i]$.

3. Complex Powers

Much of our restrictions on z and u from the previous section do not hold when we consider $z, u \in \mathbb{C}$. We will isolate z in terms of u, starting from (1.3) with

$$i^z - i^{-z} = u \tag{3.1}$$

We then make the substitution $v = i^z$ and obtain

$$\frac{v^2 - 1}{v} = u \implies v^2 - uv - 1 = 0 \tag{3.2}$$

From which it follows by the quadratic formula

$$v = \frac{1}{2}u \pm \sqrt{\frac{1}{4}u^2 + 1} \tag{3.3}$$

[1]Our bar for 'non-trivial' is quite low. It should be noted (2.8) and (2.9) are equivalent statements, as are (2.10) and (2.11). Also, (2.12) and (2.13) follow immediately from $\frac{1}{i} = -i$.

Recalling $v = i^z$, we have

$$i^z = \frac{1}{2}u \pm \sqrt{\frac{1}{4}u^2 + 1} \tag{3.4}$$

We take the logarithm base i of both sides to obtain

$$z = \log_i \left(\frac{1}{2}u \pm \sqrt{\frac{1}{4}u^2 + 1}\right) \tag{3.5}$$

We now use the change of base property $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$ and Euler's formula $e^{ix} = \cos(x) + i\sin(x)$. The latter equation evaluated at $x = \frac{\pi}{2}$ yields $e^{\pi i/2} = i$ from whence it follows, if we choose log the natural logarithm to be the principal branch of the complex logarithm, that $\ln(i) = \frac{\pi i}{2}$. Applying this with the former logarithmic identity to (3.6), we have

$$z = -\frac{2i}{\pi} \log \left(\frac{1}{2}u \pm \sqrt{\frac{1}{4}u^2 + 1}\right) \tag{3.6}$$

We may factor out a 2 from within the logarithm in (3.7) and, after some algebraic manipulation, obtain

$$z = -\frac{2i}{\pi}(\log(u \pm \sqrt{u^2 + 4}) - \log(2)) \tag{3.7}$$

Though we have found what we are looking for, we substituted (3.8) into i^z and sacrificed a small algebraic goat to find that

$$i^{-\frac{2i}{\pi}(\log(u \pm \sqrt{u^2+4}) - \log(2))} = \frac{u \pm \sqrt{u^2 + 4}}{2} \tag{3.8}$$

So, substituting the above into (1.3) we have

$$\frac{u \pm \sqrt{u^2 + 4}}{2} = u + \frac{2}{u \pm \sqrt{u^2 + 4}} \tag{3.9}$$

We may choose any u a Gaussian integer[2] and the left-hand side of the above equation will be a complex power of i described by (3.8), and it will satisfy (1.3) for u a Gaussian integer, as desired.

Remark 3.1. *On the usefulness of (3.8) and (3.9), notice that for our choice of u, we may not only determine the power of i for which (1.3) is satisfied, but also know immediately that (1.2) for $u = u_0$ converges to i^z if $|\arg(u)| < \frac{\pi}{2}$, by Van Vleck's theorem* [1].

Remark 3.2. *What we have obtained in* (3.8) *is the solution to the continued fraction functional equation*

$$f(x) = x + \frac{1}{f(x)} \tag{3.10}$$

Which may also be obtained with the quadratic formula, though the formula does not provide general solutions to the functional equation

$$f(x) = x + \frac{1}{f(x^n)} \tag{3.11}$$

With non-trivial $f : \mathbb{C} \to \mathbb{C}/\{0\}$ and n a positive integer. For lack of a better term, we say solutions to (3.11) *are simple continued fraction functions of the n-th kind, the*

[2]The equation works for any choice of u.

right-hand side of (3.8) *being the two unique solutions when* $n = 1$. *Though beyond the scope of this short note, the method used in this paper to discern the functions of the first kind may also yield solutions for greater* n, *where the quadratic formula fails.*

References

[1] Edward B. Van Vleck, *On the convergence of algebraic continued fractions whose coefficients have limiting values*, Transactions of the American Mathematical Society **5** (1904), no. 3, 253–262.