Extremal Problems for Roman Domination

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Abstract

A *Roman dominating function* of a graph G is a labeling $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The *Roman domination number* $\gamma_R(G)$ of G is the minimum of $\sum_{v\in V(G)} f(v)$ over such functions. Let G be a connected n-vertex graph. We prove that $\gamma_R(G) \le 4n/5$, and we characterize the graphs achieving equality. We obtain sharp upper and lower bounds for $\gamma_R(G)+\gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, improving known results for domination number. We prove that $\gamma_R(G) \le 8n/11$ when $\delta(G) \ge 2$ and $n \ge 9$, and this is sharp.

1 Introduction

According to [6], Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself.

The objective, of course, is to minimize the total number of legions needed. The problem generalizes to arbitrary graphs. A *Roman dominating function (RDF)* on a graph G is a vertex labeling $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. For an RDF f, let $V_i(f) = \{v \in V(G)\colon f(v) = i\}$. In the context of a fixed RDF, we suppress the argument and simply write V_0, V_1, and V_2. Since this partition determines f, we can equivalently write $f = (V_0, V_1, V_2)$. The *weight* $w(f)$ of an RDF f is $\sum_{v\in V(G)} f(v)$, which equals $|V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF of G. Thus, $\gamma_R(G)$ is the minimum number of legions needed to protect cities whose adjacency graph is G.

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Roman domination also models other facility location problems. Instead of interpreting $f(v)$ as the number of units placed at v, we can view it as a cost function. Units with cost 2 may be able to serve neighboring locations, while units with cost 1 can serve only their own location. For example, in a communication network, wireless hubs are more expensive but can serve neighboring locations, while wired hubs are low-range but are cheaper.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [6] began the study of Roman domination, suggested in a *Scientific American* article by Stewart [17] and even earlier by ReVelle [21]. Since $V_1 \cup V_2$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, [6] observed that

$$\gamma(G) \le \gamma_R(G) \le 2\gamma(G), \tag{1}$$

where $\gamma(G)$ is the domination number of G. In a sense, $2\gamma(G)-\gamma_R(G)$ measures "inefficiency" of domination, since when $\gamma_R(G) = (2-\beta)\gamma(G)$, at least the fraction β of the vertices in a minimum dominating set serve only to dominate themselves.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [6] studied basic properties of Roman dominating functions and calculated γ_R for specific graphs. They characterized the graphs G such that $\gamma_R(G) \le \gamma(G) + k$ when $k \le 2$; this was extended to larger k in [22]. They also characterized graphs G such that $\gamma_R(G) = 2\gamma(G)$ in terms of 2-packings, calling such graphs *Roman.* Henning [11] characterized Roman trees, while Song and Wang [16] characterized the trees T with $\gamma_R(T) = \gamma(T) + 3$. The computational complexity of $\gamma_R(G)$ was studied in [7]. Linear-time algorithms for computing $\gamma_R(G)$ are known on interval graphs [14, 4], cographs [14], and strongly chordal graphs [4]. A polynomial-time algorithm is known on AT-free graphs [14]. Other related domination models were studied in [5, 8, 9, 12, 13].

In this paper, we study extremal problems for $\gamma_R(G)$ on various classes of n-vertex graphs. In Section 2, we prove that $\gamma_R(G) \le 4n/5$ when G is connected and $n \ge 3$, and we determine when equality holds. In Section 3, we obtain sharp upper and lower bounds for $\gamma_R(G)+\gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, where $\overline{G}$ denotes the complement of G. We use these ideas to determine the n-vertex graphs G with largest value of $\gamma(G)\gamma(\overline{G})$, shown to equal n in [18].

Let $\delta(G)$ denote the minimum vertex degree in G. When $\delta(G) \ge k$, inequality (1) and the well-known upper bound on $\gamma(G)$ from [1, 20] yield $\gamma_R(G) \le 2\frac{1+\ln(k+1)}{k+1}n$. This was improved slightly in [6]; we use their improvement in Section 3. For small k, the optimal coefficient is of interest. In Section 4, we prove that if G is a connected n-vertex graph with $\delta(G) \ge 2$ and $n \ge 9$, then $\gamma_R(G) \le 8n/11$. The bound is sharp, and we determine when equality holds.

In an earlier version of this paper, we conjectured that $\gamma_R(G) \le \lceil 2n/3 \rceil$ for 2-connected graphs, and we proved this for graphs having spanning subgraphs consisting of some number of cycles linked in a ring by paths joining nonadjacent vertices on the cycles (these subgraphs are minimal 2-connected graphs). Subsequently, Chang and Liu [2] disproved the conjecture by constructing 2-connected n-vertex graphs such that $\gamma_R(G) = 23n/34$ for infinitely many n; note that $\frac{23}{34} = \frac{2}{3} + \frac{1}{102}$. The key graph in their construction is obtained from K_4 by replacing each edge uv with a 5-cycle C plus edges from nonadjacent vertices of C to u and

v; this graph G has 34 vertices, and $\gamma_R(G) = 23$. They also settled the problem by proving that $\gamma_R(G) \le \max\{\lceil 2n/3\rceil, 23n/34\}$ when G is 2-connected. For minimum degree 3, they proved in [3] that $\gamma_R(G) \le 2n/3$ and that this is sharp for infinitely many 3-connected graphs; see also [4] and other forthcoming papers.

Our graphs have no loops or multiple edges; we use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. The degree of a vertex v in G is $d_G(v)$ or simply $d(v)$. The minimum and maximum vertex degrees are $\delta(G)$ and $\Delta(G)$. For a set $S \subseteq V(G)$, the *(open) neighborhood* of S is $\{v \in V(G) - S\colon\ v \text{ has a neighbor in } S\}$, denoted $N(S)$. The *closed neighborhood* of S is $N(S) \cup S$, denoted $N[S]$. When $S = \{v\}$, we simply write $N(v)$ and $N[v]$. The *diameter* of G is the maximum distance between vertices of G, denoted $\operatorname{diam} G$. In a tree, a *penultimate vertex* is any neighbor of a leaf. We write P_n, C_n, and K_n for the path, cycle, and complete graph with n vertices, respectively. We write mG for the graph consisting of m disjoint copies of G.

2 Connected Graphs

For n-vertex graphs, always $\gamma_R(G) \le n$, with equality when $G = \overline{K}_n$. In this section we prove that $\gamma_R(G) \le 4n/5$ when G is a connected n-vertex graph and characterize when equality holds. Since $\gamma(G)$ may be as high as $n/2$, (1) only gives $\gamma_R(G) \le n$, so proving the bound of $4n/5$ needs additional work. Since deleting an edge cannot decrease γ_R, it suffices to prove the bound for trees.

Theorem 2.1 *If T is an n-vertex tree, with $n \ge 3$, then $\gamma_R(T) \le 4n/5$.*

Proof. We use induction on n. The base step handles trees with few vertices or small diameter. If $\operatorname{diam} T = 2$, then T has a dominating vertex, and $\gamma_R(T) \le 2 < 4n/5$. If $\operatorname{diam} T = 3$, then T has a dominating set of size 2, which yields $\gamma_R(T) \le 4$. This is sufficiently small for trees with at least six vertices. For $n \in \{4, 5\}$ and $\operatorname{diam} T = 3$, a penultimate vertex has degree 2; putting weight 2 on the other penultimate vertex and weight 1 on the undominated leaf yields $\gamma_R(T) \le 3$, which is small enough.

Hence we may assume that $\operatorname{diam} T \ge 4$. For a subtree T' with n' vertices, where $n' \ge 3$, the induction hypothesis yields an RDF f' of T' with weight at most $\frac{4}{5}n'$. We find a subtree T' such that adding a bit more weight to f' will yield a small enough RDF f for T.

Let P be a longest path in T chosen to maximize the degree of its next-to-last vertex v, and let u be the non-leaf neighbor of v.

Case 1: $d_T(v) > 2$. Obtain T' by deleting v and its leaf neighbors. Since $\operatorname{diam} T \ge 4$, we have $n' \ge 3$. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(x) = 0$ for each leaf x adjacent to v. Note that f is an RDF for T and that $w(f) = w(f') + 2 \le \frac{4}{5}(n - 3) + 2 < \frac{4}{5}n$.

Case 2: $d_T(v) = d_T(u) = 2$. Obtain T' by deleting u and v and the leaf neighbor z of v. If $n' = 2$, then T is P_5 and has an RDF of weight 4. Otherwise, the induction hypothesis applies. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(u) = f(z) = 0$. Again f is an RDF, and the computation $w(f) < \frac{4}{5}n$ is the same as in Case 1.

Case 3: $d_T(u) > 2$ *and every penultimate neighbor of* u *has degree 2.* If every neighbor of u is penultimate or a leaf, then $\operatorname{diam} T = 4$ and T is obtained from a star with center u by subdividing k edges, where $k \geq 2$. Put weight 2 on u and weight 1 on the non-neighbors of u. Now $w(f) = k + 2$ and $n \geq 2k + 1 \geq 5$, so $w(f) \leq (n + 3)/2 \leq \frac{4}{5}n$.

Otherwise, some neighbor t of u is neither penultimate nor a leaf. Obtain T' from T by deleting the vertices of the component of $T - tu$ containing u. Now $n' \geq 3$ and the induction hypothesis applies. Define f on $V(T)$ by $f(x) = f'(x)$ except for $f(u) = 2$, $f(x) = 1$ for each non-neighbor x of u outside T', and $f(x) = 0$ for $x \in N(u) - \{t\}$. Again f is an RDF. We have $w(f) = w(f') + k + 2$, where k is the number of leaves of T at distance 2 from u.

If $k = 1$, then $d_T(u) > 2$ forces u to have a leaf neighbor, and $w(f) \leq \frac{4}{5}(n - 4) + 3 < \frac{4}{5}n$. Otherwise $k \geq 2$, and $w(f) \leq \frac{4}{5}(n - 2k - 1) + (k + 2) = \frac{1}{5}(4n - 3k + 6) \leq \frac{4}{5}n$. □

As shown in [6], $\gamma_R(P_n) \leq (2n + 2)/3$. The path is not the worst-case n-vertex tree; equality in Theorem 2.1 is achievable. Let L_k consist of the disjoint union of k copies of P_5 plus a path through the central vertices of these copies, as illustrated in Figure 1.

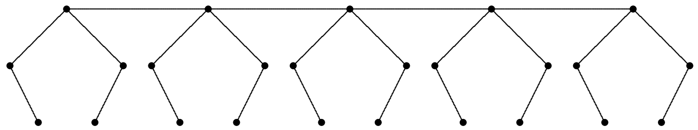


Figure 1: The tree L_5.

If u is a vertex of degree 2 having a leaf neighbor v, then an RDF must put total weight at least 2 on $\{u, v\}$ unless the other neighbor of u has weight 2. Thus when two such vertices u and u' have a common neighbor w, an RDF must give total weight at least 4 to $\{v, u, w, u', v'\}$. In L_k, there are k disjoint 5-vertex sets of this form, so $\gamma_R(L_k) \geq 4k = 4n/5$. Such copies of P_5 can be assembled in many ways, and this allows us to characterize the trees achieving equality in Theorem 2.1.

Theorem 2.2 *If* T *is an* n*-vertex tree, then* $\gamma_R(T) = 4n/5$ *if and only if* $V(T)$ *can be partitioned into sets inducing* P_5 *such that the subgraph induced by the central vertices of these paths is connected.*

Proof. We have observed that if an induced subgraph H of G is isomorphic to P_5, and its noncentral vertices have no neighbors outside H in G, then every RDF of G puts weight at least 4 on $V(H)$. Thus in any tree with such a vertex partition, weight at least 4 is needed on every set in the partition.

To show that equality requires this structure, we examine the proof of Theorem 2.1 more closely. The proof is by induction on n. In the base cases and Cases 1 and 2, we produce an RDF with weight less than $4n/5$. In Case 3 with diameter 4, equality requires $n = 2k + 1$ and $k = 2$, and the only such tree is P_5 itself.

Define u, T', n', t, k as in the inductive part of Case 3. The bound holds with equality only if $k = 2$ and $n' = n - (2k + 1)$. Thus u has no leaf neighbors, and $T - V(T')$ is a 5-vertex path Q with center u. Equality also requires $\gamma_R(T') = 4n'/5$, so by the induction hypothesis T' has the specified form. In particular, t lies in a copy P' of P_5 in a covering of $V(T')$ by 5-sets inducing paths. Let t' be the center of P'.

If $t \neq t'$, then we build a cheaper RDF for T. Put weight 2 on u and weight 1 on the leaves of Q. Put weight 1 on the neighbor of t in $T' - t'$, and put weight 2 on the penultimate vertex of P' farthest from t. We have now guarded $P' \cup Q$ using total weight 7, and hence $\gamma_R(T) < \frac{4}{5}n$. Hence equality requires $t = t'$ and the specified structure for T. $\square$

It is easy to extend this characterization to all connected graphs.

Theorem 2.3 *If G is a connected n-vertex graph, then $\gamma_R(G) \leq 4n/5$, with equality if and only if G is C_5 or is obtained from $\frac{n}{5}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{5}P_5$.*

Proof. If G has the specified form, then as remarked earlier every RDF puts weight at least 4 on the vertex set of each copy of P_5.

Now suppose that $\gamma_R(G) = \frac{4}{5}n$. Since adding edges cannot increase γ_R, every spanning tree of G has the form specified in Theorem 2.2. Given a spanning tree T, let $S_1, \ldots, S_k$ be the 5-sets in the special partition of $V(T)$. The assignment of weight 4 that guards S_i can be chosen independently of any other S_j. If any edge of G joins vertices of S_i and S_j that are not the centers of the paths they induce, then an RDF with weight less than $\frac{4}{5}n$ can be built as in the proof of Theorem 2.2. $\square$

3 Nordhaus-Gaddum Inequalities

For a graph parameter ρ, bounds on $\rho(G) + \rho(\overline{G})$ and $\rho(G)\rho(\overline{G})$ in terms of the number of vertices are called results of "Nordhaus–Gaddum" type, honoring the paper of Nordhaus and Gaddum [15] obtaining such bounds when ρ is the chromatic number.

For an n-vertex graph G with $n \geq 2$, it is known (see [10, p. 237]) that

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1 \tag{2}$$

$$2 \leq \gamma(G)\gamma(\overline{G}) \leq n. \tag{3}$$

In this section we obtain the analogous sharp results for γ_R.

Proposition 3.1 *If G is an n-vertex graph, then $\gamma_R(G) \le n - \Delta(G) + 1$.*

Proof. When v is a vertex of maximum degree, the RDF $(N(v), V(G) - N[v], \{v\})$ has weight $n - \Delta(G) + 1$. $\square$

Theorem 3.2 *If G is an n-vertex graph, with $n \ge 3$, then*

$$5 \le \gamma_R(G) + \gamma_R(\overline{G}) \le n + 3.$$

Furthermore, equality holds in the upper bound only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

Proof. When G has at least three vertices, $\gamma_R(G) \ge 2$, with equality only when G has a dominating vertex. Since a graph and its complement cannot both have dominating vertices, $\gamma_R(G) + \gamma_R(\overline{G}) \ge 5$. Equality holds if and only if G or $\overline{G}$ has a vertex of degree $n-1$ and its complement has a vertex of degree $n-2$.

For the upper bound, Proposition 3.1 yields

$$\begin{aligned}\gamma_R(G) + \gamma_R(\overline{G}) &\le (n - \Delta(G) + 1) + (n - \Delta(\overline{G}) + 1)\\ &= n - \Delta(G) + \delta(G) + 3 \le n + 3.\end{aligned}$$

If $\gamma_R(G) + \gamma_R(\overline{G}) = n + 3$, then equality holds throughout the calculation, and $\delta(G) = \Delta(G)$. Hence G is k-regular for some k. We may assume that $k \le (n-1)/2$, since the argument is symmetric in G and $\overline{G}$. Since equality holds, $\gamma_R(G) = n-k+1$ and $\gamma_R(\overline{G}) = k+2$.

Let $v \in V(G)$. If some vertex u outside $N[v]$ has at least two neighbors outside $N[v]$, then the RDF $(N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ has weight at most $n-k$, a contradiction. Hence every vertex not in $N[v]$ has at least $k-1$ neighbors in $N(v)$. Similarly, each vertex in $N(v)$ has at most two neighbors outside $N[v]$.

Counting the edges joining $N(v)$ and $V(G) - N[v]$ from both sides yields $(k-1)(n-k-1) \le 2k$, simplifying to $n \le k + 3 + \frac{2}{k-1}$ for $k > 1$. Since $n \ge 2k + 1$, we have $k \le 2 + \frac{2}{k-1}$, which requires $k \le 3$. If $k = 3$, then $n = 7$, but there is no 3-regular 7-vertex graph.

For $k = 2$, we have $n \le k + 3 + \frac{2}{k-1} = 7$ and $n \ge 2k + 1 = 5$. For each 2-regular graph G with $n \in \{6, 7\}$, we have $\gamma_R(G) = n - 2$, so $\gamma_R(G) = n - k + 1$ leaves only $G = C_5$.

For $k = 1$, the only example is $\frac{n}{2}K_2$, where equality holds. For $k = 0$, the only example is $G = \overline{K}_n$, where $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$, and equality does not hold. $\square$

For the product bound, (1) and (3) yield $\gamma_R(G)\gamma_R(\overline{G}) \le 4n$. The optimal bound is smaller for sufficiently large n. We will prove in Theorem 3.4 that $\gamma_R(G)\gamma_R(\overline{G}) \le 16n/5$ when $n \ge 160$. Sharpness is shown by $G = kC_5$, since $\gamma_R(kC_5) = 4k$ and $\gamma_R(\overline{kC_5}) = 4$ and $|V(kC_5)| = 5k$. In fact, equality holds only when G or $\overline{G}$ is kC_5 (when n is large).

The most difficult case in the proof of Theorem 3.4 is when $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$. We handle this case separately in the next lemma, using a result from Cockayne, Dreyer, Hedetniemi, and Hedetniemi [6]. For an n-vertex graph G, they proved that

$$\gamma_R(G) \le \frac{2 + 2\ln((1+\delta(G))/2)}{1+\delta(G)} n. \tag{4}$$

Since $\gamma_R(G) \le 2\gamma(G)$, this bound slightly refines the well-known bound $\gamma(G) \le \frac{1+\ln(1+\delta(G))}{1+\delta(G)} n$ due to Arnautov [1] and Payan [20].

Lemma 3.3 *If G is an n-vertex graph with $n \ge 160$, and $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.*

Proof. Let G be such a graph, and let v be a vertex of minimum degree in G. If $d(v) \le 2$, then the diameter constraint implies that $(V(G) - N(v), \varnothing, N(v))$ is an RDF of G and that $(V(G) - N[v], N(v), v)$ is an RDF of $\overline{G}$, so $\gamma_R(G)\gamma_R(\overline{G}) \le 16$. Hence we may assume that $d_G(v) \ge 3$, and similarly $\delta(\overline{G}) \ge 3$.

Let $R = V(G) - N_G[v]$. We choose a family of disjoint subsets of $N_G(v)$ dominating R as follows. Initialize $B_1 = N_G(v)$; note that B_1 dominates R, since $\operatorname{diam} G = 2$. If B_i dominates R, then let A_i be a minimal subset of B_i dominating R, and let $B_{i+1} = B_i - A_i$. If B_{i+1} does not dominate R, then stop, setting $q = i$ and $A^* = B_q$. Otherwise, increment i. Note that $A_1, \ldots, A_q$ partition $N_G(v) - A^*$, with each A_i being a minimal set that dominates R.

Since A_i is a minimal set that dominates R, there is a vertex $r_i \in R$ having only one neighbor in A_i; let a_i be this neighbor. Since A^* does not dominate R, there exists $w \in R$ such that $A^* \subseteq N_{\overline{G}}(w)$. Let $S = \{r_1, \ldots, r_q\} \cup \{v, w\}$ and $T = \{a_1, \ldots, a_q\}$. Now $(V(G) - (S \cup T), T, S)$ is an RDF for $\overline{G}$, since v dominates R, w dominates A^*, and r_i dominates $A_i - \{a_i\}$. Thus $\gamma_R(\overline{G}) \le 3q + 4$, which reduces to $3q + 2$ if $A^* = \varnothing$.

Let $U = A_j \cup \{v\}$, where $|A_j| = \min_i |A_i|$. Note that U is a dominating set of G. If $|U| = 2$, then $\gamma_R(G) \le 4$. Since $\overline{G}$ is connected and $\delta(\overline{G}) \ge 3$, Theorem 2.3 yields $\gamma_R(\overline{G}) < 4n/5$. Hence we may assume that $|U| > 2$, which requires $q \le \delta(G)/2$.

If $q = 1$, then $\gamma_R(\overline{G}) \le 7$ and $\gamma_R(G) \le 2|U| \le 2(\delta(G)+1)$, so $\gamma_R(G)\gamma_R(\overline{G}) \le 14(\delta(G)+1)$. Hence we may assume in this case that $\delta(G) \ge 8n/35 - 1$, but now (4) yields $\gamma_R(G) \le \frac{1+\ln(4n/35)}{4/35}$. Since $7 \cdot \frac{1+\ln(4n/35)}{4/35} < \frac{16n}{5}$ when $n \ge 54$, we have $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.

Hence we may assume that $2 \le q \le \delta(G)/2$. Using the RDF $(V(D) - U, \varnothing, U)$ and maximizing over $2 \le q \le \delta(G)/2$ (which requires $\delta(G) \ge 4$) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \le \left(\frac{2\delta(G)}{q} + 2\right)(3q+4) = (6\delta(G) + 8) + \left(6q + \frac{8\delta(G)}{q}\right) \le 10\delta(G) + 20. \tag{5}$$

Since $10\delta(G) + 20 < 16n/5$ when $\delta(G) + 2 < 8n/25$, we may assume that $\delta(G) \ge 8n/25 - 2$, and similarly for $\delta(\overline{G})$. By (4), $\max\{\gamma_R(G), \gamma_R(\overline{G})\} \le \frac{2+2\ln(4n/25-1/2)}{8n/25-1} n$. With $n \ge 160$, this bound is less than $16n/95$.

If $q \le 5$, then $\gamma_R(\overline{G}) \le 19$. If $q \ge \delta(G)/8$, then $\gamma_R(G) \le 18$. In these cases we obtain $\gamma_R(G)\gamma_R(\overline{G}) < \frac{16n}{95} \cdot 19 = 16n/5$.

Hence we may assume that $6 \le q \le \delta(G)/8$. Now $(2\delta(G)/q+2)(3q+4) \le 22\delta(G)/3+44$, since $\delta(G) \ge 48$. This bound is less than $16n/5$ when $\delta(G) < 24n/55-6$, so we may assume that $\delta(G)$ and $\delta(\overline{G})$ are at least $24n/55-6$. Now (4) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \le \left(\frac{(2+2\ln(12n/55))n}{24n/55-5}\right)^2.$$

The upper bound is less than $16n/5$ when $n \ge 160$. □

The proof actually yields $\gamma_R(G)\gamma_R(\overline{G}) = O((n\ln n)^{2/3})$ when $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$. The first part of the proof yields a bound that is linear in d, where $d = \min\{\delta(G), \delta(\overline{G})\}$, while the Arnautov–Payan bound yields a bound of the form $O([(n\ln d)/d]^2)$. The minimum of the two bounds is largest when d grows like $(n\ln n)^{2/3}$, so the bound is always $O((n\ln n)^{2/3})$.

Theorem 3.4 *If G is an n-vertex graph and $n \ge 160$, then*

$$\gamma_R(G)\gamma_R(\overline{G}) \le \frac{16n}{5},$$

with equality only when G or $\overline{G}$ is $\frac{n}{5}C_5$.

Proof. If G has an isolated vertex or edge, then $\gamma_R(\overline{G}) \le 3$, which yields $\gamma_R(G)\gamma_R(\overline{G}) \le 3n < 16n/5$. Thus we may assume that each component of G has at least three vertices. Applying Theorem 2.1 to each component now yields $\gamma_R(G) \le 4n/5$.

If $\operatorname{diam} G \ge 3$, then G has vertices u and v with no common neighbor. Hence $\{u, v\}$ is a dominating set in $\overline{G}$, and $\gamma_R(\overline{G}) \le 4$. Thus $\gamma_R(G)\gamma_R(\overline{G}) \le (4n/5)4$ when $\operatorname{diam} G \ge 3$, and similarly when $\operatorname{diam} \overline{G} \ge 3$. Lemma 3.3 produces the desired bound in the remaining case.

Since Lemma 3.3 establishes strict inequality, the only way to achieve equality in this bound is if $\gamma_R(G) = 4n/5$ and $\gamma_R(\overline{G}) = 4$ (or vice versa). If $\gamma_R(\overline{G}) = 4$, then $\delta(G) \ge 2$, so Theorem 2.3 implies that every component of G is a 5-cycle. □

A similar analysis gives the analogous result for domination number.

Theorem 3.5 *If G is an n-vertex graph, with $n \ge 184$, then equality holds in the bound $\gamma(G)\gamma(\overline{G}) \le n$ of* (3) *if and only if $\gamma(G)$ or $\gamma(\overline{G})$ equals n or $n/2$.*

Proof. If G or $\overline{G}$ is K_n, then equality holds.

If $\delta(G) = 1$, then $\gamma(\overline{G}) = 2$, and equality holds if and only if $\gamma(G) = n/2$. It is known (see [10]) that an n-vertex graph G without isolated vertices has domination number $n/2$ if and only if $G = C_4$ or G is obtained from some graph with $n/2$ vertices by adding a pendant edge to each vertex. Thus if $n > 4$ and $\gamma(G) = n/2$, then $\gamma(G)\gamma(\overline{G}) = n$.

For $\delta(G) \geq 2$, McQuaig and Shepherd [19] proved that $\gamma(G) \leq 2n/5$. If also $\operatorname{diam} \overline{G} \geq 3$, then $\gamma(G)\gamma(\overline{G}) \leq 4n/5 < n$. Hence we may assume that both G and $\overline{G}$ have diameter 2.

When $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, essentially the same argument (with obvious changes) as in the proof of Lemma 3.3 shows that $\gamma(G)\gamma(\overline{G}) < n$ for $n \geq 184$. We omit the details. □

4 Minimum Degree 2

In this section, we consider how large γ_R can be for connected n-vertex graphs with minimum degree at least 2. In the n-vertex graph G illustrated in Figure 2, an RDF must give weight 4 to an induced 5-cycle unless one of its vertices has an outside neighbor with weight 2. When there is one such vertex, deleting it from the 5-cycle leaves a 4-vertex path that still needs weight 3 on it to be guarded. Hence each subgraph formed from two 5-cycles and a common neighbor must receive weight at least 8, and we obtain $\gamma_R(G) = 8n/11$.

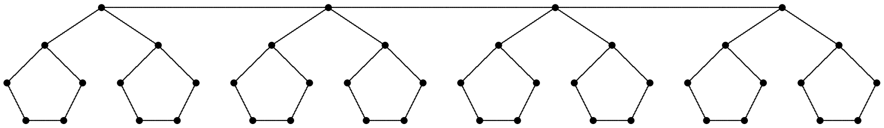


Figure 2: n-vertex graph G with $\gamma_R(G) = 8n/11$.

Lemma 4.1 *Let G be a graph with $\delta(G) \geq 2$. If G contains any configuration listed below, then there exists G' such that $\delta(G') \geq 2$, $|V(G')| \leq |V(G)| - 3$, and $\gamma_R(G) \leq \gamma_R(G') + 2$.*
a) An induced 5-vertex path P whose internal vertices have degree 2 in G.
b) Two nonadjacent vertices x and y that have at least two common neighbors with degree 2 in G and each have an additional neighbor.
c) An induced 6-cycle C with exactly two vertices having degree at least 3 in G.

Proof. In each case, we define a graph G' with at most $|V(G)| - 3$ vertices such that $\delta(G') \geq 2$, let f' be an RDF of G', and produce an RDF f of G with $w(f) \leq w(f') + 2$.

(a) Let the vertices of P be x, u, v, w, y in order. Since C is an induced path, x and y are neither equal nor adjacent. Form G' from G by deleting $\{u, v, w\}$ and adding the edge xy; every vertex of G' has the same degree in G' as in G. Let $f(v) = 2$ and $f(u) = f(w) = 0$, with $f(z) = f'(z)$ for $z \in V(G')$. This suffices unless $\{f'(x), f'(y)\} = \{2, 0\}$ and the edge xy is needed for f' to be an RDF. By symmetry, we may assume $f'(y) = 0$; in this case, let $f(w) = 2$ instead of $f(v) = 2$.

(b) Let S be the set of common neighbors of x and y with degree 2. Form G' by contracting all edges incident to S; this merges x and y into a single vertex v. Since x and

y each have a neighbor outside S, we have $d_{G'}(v) \geq 2$ and $\delta(G') \geq 2$. For $z \in V(G') - \{v\}$, let $f(z) = f'(z)$. If $f'(v) \in \{1, 2\}$, then let $f(x) = f'(v)$, $f(y) = 2$, and $f(z) = 0$ for $z \in S$. If $f'(v) = 0$, then f' puts weight 2 on a neighbor of x or y, say x; let $f(y) = 2$ and $f(x) = f(z) = 0$ for $z \in S$.

(c) If x and y are not opposite on C, then case (a) applies. Otherwise, form G' by contracting C into a single vertex v and adding a 3-cycle C' through v and two new vertices. An RDF f' of G' must put total weight at least 2 on $V(C')$. Let $f(x) = f(y) = 2$, put weight 0 on $V(C) - \{x, y\}$, and let $f(z) = f'(z)$ for $z \in V(G) - V(C)$.

In each case, $w(f) \leq w(f') + 2$. □

A *spider* is a tree consisting of at least three paths having a common endpoint. The common endpoint is the only vertex of degree at least 3 in the spider and is its *branchpoint*. A spider is completely specified by listing the distances of the leaves from the branchpoint.

Lemma 4.2 *If G is an n-vertex spider with branchpoint v, then $\gamma_R(G) \leq 8n/11$ unless $d(v) = 3$ and the leaves have distances $(1, 3, 3)$ or $(2, 2, 3)$ from v. Among the remaining spiders, $\gamma_R(G) < 8n/11$ unless $d(v) = 4$ and the leaves have distances $(1, 3, 3, 3)$ or $(2, 2, 3, 3)$ from v, or $d(v) = 3$ and the leaf distances from v are obtained from $(1, 3, 3)$ or $(2, 2, 3)$ by adding 3 to one coordinate.*

Proof. Let l_i be the number of leaves at distance i from v. Suppose first that the longest path from v has length at most 3, so $n = 1 + l_1 + 2l_2 + 3l_3$. For any path of length 3 from v, f puts weight 2 on the penultimate vertex and weight 0 on the others.

If $l_1 = l_2 = 0$, then $l_3 \geq 3$. Complete the RDF f by $f(v) = 1$. Now $w(f) = 1 + 2l_3$, and $1 + 2l_3 < \frac{8}{11}(1 + 3l_3)$ when $l_3 \geq 2$.

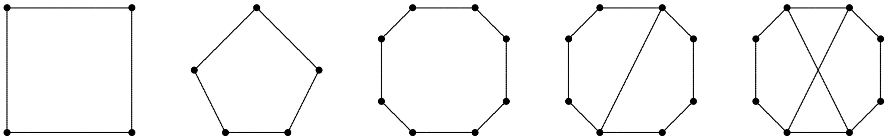
If $l_1 = 0$ and $l_2 = 1$, then put weight 2 on the neighbor of v along the short path, and let $f(v) = 0$. Now $w(f) = 2 + 2l_3$, and $2 + 2l_3 < \frac{8}{11}(3 + 3l_3)$ when $l_3 \geq 0$.

Otherwise, let $f(v) = 2$ and put weight 1 on leaves at distance 2 from v to complete the RDF f. Now $w(f) = 2 + l_2 + 2l_3$. We seek $2 + l_2 + 2l_3 < \frac{8}{11}(1 + l_1 + 2l_2 + 3l_3)$, which is equivalent to $14 < 8l_1 + 5l_2 + 2l_3$. Since we have $l_1 + l_2 + l_3 \geq 3$ and $l_1 + l_2 \geq 1$ with equality in the latter only when $l_1 = 1$, the right side is at least 15 except in four cases. For $(l_1, l_2, l_3) \in \{(1, 0, 2), (0, 2, 1)\}$ the right side is 12, and we have $n = 8$ and $\gamma_R(G) = 6$. For $(l_1, l_2, l_3) \in \{(1, 0, 3), (0, 2, 2)\}$ the right side is 14, and we have $n = 11$ and $\gamma_R(G) = 8$.

With the spiders above as a basis, we now apply induction on n. We may assume that G has some path of length more than 3 from v. Let G' be the graph obtained from G by deleting three vertices from the end of a longest such path. Using weight 2 on the middle of those three vertices yields $w(G) \leq w(G') + 2$. Since $2/3 < 8/11$, the induction hypothesis yields $\gamma_R(G) < 8n/11$ unless G' is one of the two 8-vertex spiders that fail the bound. In this case, $n = 11$ and $\gamma_R(G) \leq 8$, so the desired ratio holds with equality. □

A *thread* in a graph G is a trail whose internal vertices have degree 2 in G and whose endpoints do not have degree 2. If the endpoints of a thread are equal, then the thread is a cycle having one vertex of degree greater than 2. In a connected graph with maximum degree at least 3, the threads partition the edge set.

Theorem 4.3 *If G is a connected n-vertex graph with $\delta(G) \geq 2$ other than those shown below, then $\gamma_R(G) \leq 8n/11$.*



Proof. Note that $\gamma_R(C_4) = 3 > \frac{32}{11}$, $\gamma_R(C_5) = 4 > \frac{40}{11}$, and $\gamma_R(C_8) = 6 > \frac{64}{11}$. Also, one or two chords added to C_8 as shown above do not reduce γ_R. For each graph G shown above, $\frac{8|V(G)|}{11} < \gamma_R(G) \leq \frac{8|V(G)|}{11} + \frac{4}{11}$.

To prove the upper bound for all other graphs, we use induction on n. If G is a cycle, then the claim holds ($\gamma_R(C_7) = 5 < \frac{56}{11}$ and $\gamma_R(C_{11}) = 8$), so we may assume that $\Delta(G) \geq 3$. Our aim is to find a spanning subgraph of G in which one component G_1 is a spider to which we can apply Lemma 4.2, and the remainder G_2 is a graph to which we can apply the induction hypothesis. First we use the induction hypothesis to restrict the structure of G.

Since $2/3 < 8/11$, Lemma 4.1(a) allows us to assume that G has no induced path with at least three internal vertices of degree 2.

Since deleting an edge cannot reduce γ_R, we may assume that every edge joining two vertices with degree at least 3 is a cut-edge. In particular, no cycle in G has a chord. If G has a cut-edge uv with endpoints of degree at least 3, then let H_u and H_v be the components of $G - uv$ containing u and v, respectively. Both H_u and H_v are edge-minimal connected graphs with minimum degree at least 2.

Let $\mathcal{C} = \{C_4, C_5, C_8\}$. If neither H_u nor H_v lies in $\mathcal{C}$, then the RDFs guaranteed for them by the induction hypothesis combine to form the desired RDF of G. If $H_u, H_v \in \mathcal{C}$, then in each case weight 2 on u permits saving one unit on H_v, so

$$\gamma_R(G) \leq \gamma_R(H_u) + \gamma_R(H_v) - 1 \leq \frac{8|V(H_u)| + 4}{11} + \frac{8|V(H_v)| + 4}{11} - 1 < \frac{8n}{11}.$$

Thus when G has a cut-edge uv with $d_G(u), d_G(v) \geq 3$, we may assume that exactly one of $\{H_u, H_v\}$ lies in $\mathcal{C}$.

Similarly, if G consists of two graphs $H_u, H_v \in \mathcal{C}$ joined by a thread P having endpoints u and v plus one or two internal vertices, then H_u and H_v have optimal RDFs assigning weight 2 to u and v; together they form an RDF of G. Hence

$$\gamma_R(G) \leq \gamma_R(H_u) + \gamma_R(H_v) \leq \frac{8|V(H_u)| + 4}{11} + \frac{8|V(H_v)| + 4}{11} \leq \frac{8n}{11}.$$

Now let v be a vertex of degree at least 3 that does not lie in a member of $\mathcal{C}$ joined to the rest of G by one cut-edge. The arguments above imply that at least one end of every thread is such a vertex. We seek a subgraph G_1 consisting of $d(v)$ paths from v whose lengths do not equal 3, such that $\delta(G - V(G_1)) \geq 2$ and no component of $G - V(G_1)$ lies in $\mathcal{C}$. By Lemma 4.2 and the induction hypothesis, such a subgraph completes the proof.

Consider the threads emanating from v. If v lies on a cycle C whose other vertices have degree 2, then regardless of the length of C, it is possible to delete one edge e of C so that $C - e$ consists of two threads from v with neither having length 3.

All other threads from v lead to vertices of degree at least 3 other than v and have length at most 3 (by Lemma 4.1(a)). Let u be such a vertex, reached by a thread P with last edge e. In $G - e$, let H be the component containing u. If H is a cycle, then cutting an edge e' of H incident to u leaves $P \cup H - e'$ as a thread leaving v; we put it in G_1. The thread has length at least four unless P has length 1 and H is a 3-cycle, but then uv is a cut-edge whose deletion from G leaves two components not in $\mathcal{C}$.

If H is not a cycle, then deleting e yields a thread of length at most 2 leaving v (since P has length at most 3). However, cutting two threads that reach u from v could leave u with insufficient degree. If at least two threads reach u, then by Lemma 4.1(b,c) we may assume that exactly one thread P of length 2 and one thread P' of length 3 reach u from v.

If $d(u) \geq 4$, then we can cut each final edge. If $d(u) = 3$, then a third thread Q leaves u, ending at w. If w is not the end of another thread from v, or if $d(w) \geq 4$, then since P and P' have different lengths, we can cut the last edge of one of them so that the resulting thread from v formed by cutting the end of Q incident to w does not have length 3.

If w is the end of exactly one other thread from v in G and $d(w) = 3$, then we cut the last edge of P. Since P' has length 3, it now extends to reach w with length at least 4. When we cut the last edge of the other thread from v to w, the thread along P' and Q becomes even longer. The process can continue when v has large degree, yielding one long thread and many short threads.

If the process reaches some w' that is the end of two threads from v, and $d(w') = 3$, then cutting the edge reaching w' leaves a 5-cycle through v whose other vertices have degree 2 (the union of those two threads), and we can cut one edge of it to obtain two short threads.

In the remaining spanning subgraph, the component G_1 containing v is a union of $d(v)$ threads, none having length 3, and every other component has minimum degree at least 2 and is not one of the excluded subgraphs. As remarked above, Lemma 4.2 and the induction hypothesis now provide the desired RDF. $\square$

To characterize equality in Theorem 4.3, we study its proof closely.

Theorem 4.4 *Let F be the graph of Figure 3. Let G be a connected graph of order n with minimum degree at least 2. If $n \geq 9$, then $\gamma_R(G) = 8n/11$ if and only if*
(1) $n = 11$ and G is isomorphic to F plus a subset of one of $\{y_1y_3, y_1y_4, y_2y_3, y_2y_4\}$, $\{wz_1, y_1y_3, y_1y_4\}$, or $\{wz_1, wz_3, y_1y_3\}$ added as edges, or
(2) $n > 11$ and G consists of disjoint copies of the graphs F, $F + wz_1$, and $F + wz_1 + wz_3$ with additional edges connecting copies of w.

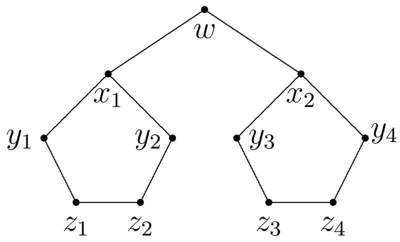


Figure 3: The graph F.

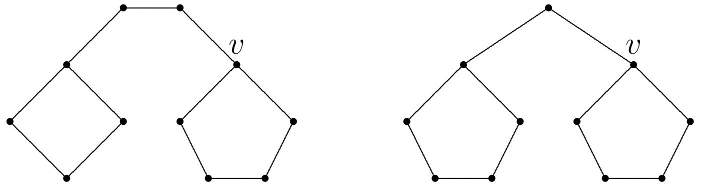
Proof. If G has the indicated form, then, regardless of the edges between copies of w, any RDF must put weight at least 8 on every copy of F, so $\gamma_R(G) \geq 8n/11$.

For the converse, let G be a graph achieving equality in Theorem 4.3. Since $2/3 < 8/11$, G cannot contain a configuration as described in Lemma 4.1. Also the deletion of any cut-edge joining vertices of degree at least 3 without leaving a component in $\mathcal{C}$ must leave components where equality holds.

Let G' be the subgraph resulting from such deletions (called G in Theorem 4.3). Let v be a vertex of G' as chosen in that proof. Since equality holds for G', it must also hold for the subgraphs G_1 and $G' - V(G_1)$ obtained in the inductive proof.

A closer look at Lemma 4.2 characterizes the vectors of path lengths where $\gamma_R(G_1) = 8|V(G_1)|/11$ can hold. Since the proof of Theorem 4.3 extracts a graph G_1 in which no thread from v has length 3, equality requires the threads from v to have lengths 2, 2, and 6.

To obtain a thread of length 6 without obtaining a thread of length 1, we must have had $d(v) = 3$, and one thread from v reaches a cycle in $\mathcal{C}$. If $n = 11$, then the possibilities are as shown below, but the graph on the left has an RDF of weight 7. Inspection shows that the only graphs with Roman domination number 8 spanned by F are those claimed.



When $n > 11$, we claim that the endpoints of the threads of length 2 from v are still adjacent and have degree 2. If not, then they would have degree at least 3, and using one of them in place of v would yield a spider as G_1 that has a thread of length 1 (by cutting the edge of the thread to v). We would then have $\gamma_R(G') < 8n/11$.

We conclude that successively deleting edges of G with endpoints of degree at least 3, without introducing components in $\mathcal{C}$, yields a graph whose components are copies of F. Since there exist minimum weight RDFs of F putting weight 2 on any given vertex, and deletion of any vertex of F other than w leaves a subgraph where weight 7 suffices, every edge of G not contained among the vertices of a single copy of F joins copies of w.

If any edge of G connects the two 5-cycles in one copy F' of F, then since G is connected, the central vertex w' of F' has a neighbor in another copy of F that can be given weight 2.

With w' protected, we can protect the rest of F' with weight 7 using the edge joining the two 5-cycles. This yields $\gamma_R(G) \leq 7 + 8(n-11)/11 < 8n/11$. Hence no edges can be added between or within the copies of F other than those described in the statement. $\square$

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