BC 3 Name:___________________________________

Upper Bounds

Consider $\sum a_n$ with $a_n \geq 0$.

Then $\{S_n\}$ is a non-decreasing sequence. Why?

If we consider this sequence $\{S_n\}$, where we have

$$S_1 \le S_2 \le S_3 \le \cdots \le S_n \le S_{n+1} \le \cdots ,$$

either this sequence will increase without bound OR it will be bounded above.

If we can find a number M such that if $S_n \leq M$ for all n, then we call M an upper bound of the sequence.

If a sequence is non-decreasing and it is bounded above, then the sequence will converge.

Example: Let $S_n = \dfrac{2n}{3n+1}$.

Find an upper bound of the sequence $\{S_n\}$.

State three other upper bounds for this sequence.

Which of these do you think is the most significant?

Why? In other words, what makes it special?

Theorem: Let $\{S_n\}$ be a non-decreasing sequence.

Either (1) If an upper bound exists, then there is a Least Upper Bound L and the sequence converges to L.

Or (2) The sequence diverges to $+\infty$. (This means the S_n eventually exceeds every given finite M.)

Now consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. While the values of a_n approach 0 as n increases, it is not as obvious whether the series converges or diverges. Use technology to plot some terms of the sequence $\{S_n\}$. Any guesses about the convergence?

Theorem: The harmonic series diverges.

Proof: We need to show that the sequence of partial sums $\{S_n\}$ increases without bound. This can be done by showing that the value of S_n can be made arbitrarily large by taking an appropriate value of n.

Consider the following:

$$S_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$S_4 = S_2$ + ______ + _____ > S_2 + ________ + ______ $= S_2 + \frac{1}{2} > \frac{3}{2}$

$S_8 = S_4$ + ______ + _____ + _____ + ______ > S_4 + ______ + _______ + ______ + ___________

$$= S_4 + \frac{1}{2} > \frac{3}{2} + \frac{1}{2} = \frac{4}{2}$$

S_{16} = ... > . . . . $= S_8$ + ________ > ___________

⋮

⋮

S_{2n} > ______________

Since this last expression (written in the blank above) is unbounded as $n \to \infty$, and since S_{2n} will always be larger, the sequence of partial sums increases without bound. Hence, the harmonic series diverges.