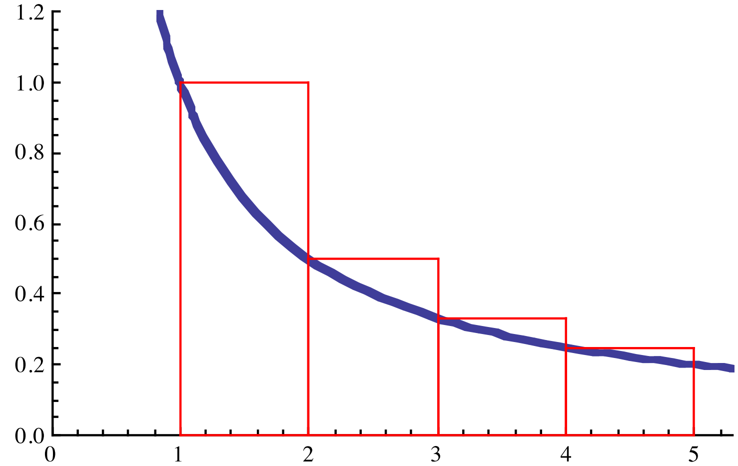
BC 3 Name:_________________________________

Integral Test for Series

Consider the series $\sum_{k=1}^{\infty} a_k$ where $a_k = 1/k$, and consider the related function $a(x) = 1/x$. Look at the integral, the graph, the endpoints, and the left-hand Riemann approximation shown below.

$$\int_1^5 a(x)\,dx = \int_1^5 \frac{1}{x}\,dx$$



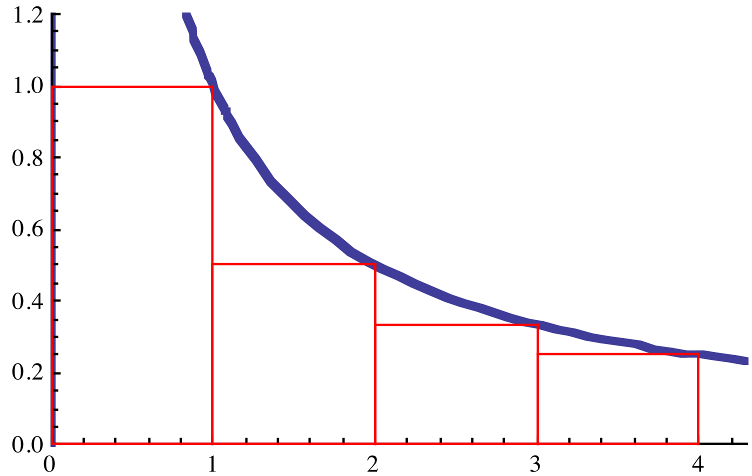
(1) Find the area of each rectangle in the graph above. (Watch the scale.) Which partial sum of the series $\sum_{k=1}^{\infty} a(k)$ is in question here? (S-sub-what?)

(2) What is the upper bound of the integral in question? Why is this value different from the n of the nth partial sum from (1)?

(3) Set up an inequality involving the sum of the areas of the rectangles and the integral.

(4) Look at the graph below. This includes one rectangle on the left and then a right-hand Riemann approximation. Find and mark the area of each rectangle.

$$1+\int_1^4 a(x)dx = 1+\int_1^4 \frac{1}{x}dx$$



(5) In the statement above with the integral sign, where did the initial "1" come from? Why must this be added to the integral?

(6) Set up an inequality involving the sum of the areas of the four rectangles and the integral expression.

(7) Use both of your inequalities from (3) and (6) above to make a 3-part inequality.

(8) Generalize your 3-part inequality in (7) for any sum $\sum_{k=1}^{n} a_k$ and any value n.

(9) Take the limit of everything in sight as $n \to \infty$.

What does this say about the series in question? (Consider two cases.)

(a) If $\int_1^{\infty} a(x)\,dx$ converges, then

(b) If $\int_1^{\infty} a(x)\,dx$ diverges, then

(10) These inequalities hold for our example $a_k = \frac{1}{k}$. What assumptions on your function must hold to be sure that these relationships (in part (8), above) still hold for an arbitrary sequence a_k?

Note that when the three parts of the inequality are put together, this implies that the integral and the related series both converge or both diverge. That is, the work that we did for improper integrals also applies to infinite series.