Back to \( f(x) = \cos x \) ...
In class, we found a polynomial that can be extended to give

\[
P(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\]

Let \( P_n(x) \) refer to the polynomial approximating the function \( f \) which has "order of contact \( n \)" at a point \( x = a \). That is to say, \( P_n^{(k)}(a) = f^{(k)}(a) \) for \( k = 0, 1, 2, 3, \ldots, n \), and here we have \( a = 0 \).

For example, with \( f(x) = \cos x \), we have

\[
P_4(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{and} \quad P_7(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}
\]

Note that "\( n \)" can cause problems. Does it mean the \( n \)th term? the order of contact? the value of \( n \) in the summation? Be careful!

(1) If we choose \( P_4(x) \), giving us 3 non-zero terms, evaluate \( f(x) \) and \( P_4(x) \) for the following values of \( x = a \).

\[
f(a) \quad g(a) = P_4(a)
\]

\[
a = 2
\]

\[
a = 1
\]

\[
a = 0.5
\]

For roughly what values of \( x \) does \( P_4 \) seem to be a good approximation of \( f \)?

(2) Again using 3 non-zero terms, let \( x = 1 \). We want to find an upper bound for the error.

(a) First, find the actual error (according to your calculator) by evaluating \(| \cos 1 - P_4(1) |\).

(b) Use the alternating series error approximation to find an upper bound for the error.

(3) Still using 3 terms of \( P \), what are the possibilities for \( x \) if the error is to be less than 0.00005?
Set the window on the calculator so that \(-7 \leq x \leq 7\) and \(-2 \leq y \leq 2\).

(4) Sketch the graphs of \(P_4(x)\) and \(f(x)\) below.

For what values of \(x\) does \(P_4\) seem to be a fairly good approximation of \(f\) ?

(5) Sketch the graphs of \(P_8(x)\) and \(f(x)\) below.

For what values of \(x\) does \(P_8\) seem to be a fairly good approximation of \(f\) ?

(6) Sketch the graphs of \(P_{12}(x)\) and \(f(x)\) below.

For what values of \(x\) does \(P_{12}\) seem to be a fairly good approximation of \(f\) ?
(7) Based on these graphs, can you make any guesses about the values of \( x \) for which the infinite series \( P \) converges?

Do you think that there is a value of \( n \) such that \( P_n(x) \) will have an error less than 0.0001 for all \( x \) in the interval \(-1000 \leq x \leq 1000\)? Why or why not?

(8) Determine the values of \( x \) for which the infinite series converges by using an appropriate test.
(9) Compare this result for \( y = \cos(x) \) to the intervals found for \( y = \ln(1 + x) \) and \( y = \tan^{-1}x \).

Can you think of any possible explanations for this distinction?

More on error analysis...

(10) Using \( P_{14}(x) \) and \( x = 6 \), find an upper bound for the error.

(11) Again using \( P_{14}(x) \), what are the possibilities for \( x \) if the error is to be less than 0.0005?

(12) If \( x = 4 \), how many terms are necessary to be sure that the error is less than .01?