2016

Series 07

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Series 7 – Lagrange Error

The Lagrange error bound for Taylor series states the following:

$$\left|\text{error}\right| \le \frac{K_{n+1}\cdot\left|x-a\right|^{n+1}}{(n+1)!}, \text{ where } \left|f^{(n+1)}(x)\right| \le K_{n+1}$$

So where does this come from? Here are some clues.
Here, let $n = 2$ and let $a = 0$. Assume $x > 0$. (The other case is similar.) By our notation, we'll assume $\left|f^{(3)}(x)\right| \le K_3$, so that

$$-K_3 \quad \le \quad f^{(3)}(t) \quad \le \quad K_3$$

Take the integral of each part of the inequality from 0 → x.

Replace all of your x's with t's. (It really is better math!)

Once again, integrate each term in your inequality from 0 → x.

Replace all of your x's with t's. (It's still better math.)

Once again, integrate each term in your inequality from 0 → x.

In the middle of your three-part inequality, if you made it this far, you should have:

$$f(x) - f(0) - f'(0)\cdot x - f''(0)\cdot\frac{x^2}{2}. \text{ This is } f(x) - P_2(x).$$

Now, use absolute value to rewrite the three-part inequality:

$$\left|f(x) - P_2(x)\right| \le \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad \text{(Yea!)}$$

More on the Lagrange error bound

A slightly different form: | error | = $\left|R_n\right| = \left|f(x) - P_n(x)\right| = \left|\dfrac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}\right|$ for some c between a and x. Given a value of n and a value of x, the remaining issue is to find an upper bound for the value of $\left|f^{(n+1)}(c)\right|$, given as K_{n+1}. In some cases, one may take into account restrictions on the value of c and then work to find an upper bound. In other cases, we simply look at the size of the $(n + 1)^{st}$ derivative for all values of x, sometimes referred to as K_{n+1}.

(1) A function f has derivatives of all orders at $x = 0$ such that $\left|f^{(n)}(x)\right| \le \dfrac{n}{4}$ for all x. Find an upper bound for the size of the error when using the Maclaurin polynomial with $x = 1.2$ and $n = 6$.

(2) A function f has derivatives of all orders at $x = 2$ such that $\left|f^{(n)}(x)\right| \le n^2$ for all x. If $x = 1.2$, find an upper bound for the size of the error when the function f is approximated by the 4^{th} degree Taylor polynomial at $x = 2$.

(3) Look back at worksheet Series 1 where we found the Maclaurin series for $f(x) = \ln(1 + x)$. At that time, we used values of $x > 0$ so that the series was alternating. For values of $x < 0$, the series no longer alternates, and we must look at the Lagrange bound. If $x = –0.3$ and $n = 3$, find an upper bound for the size of the fourth derivative by graphing this function over the interval [–0.3, 0]. Then use this maximum size to find an upper bound for the error.