On a Computer-Aided Decomposition of the Complete Digraph into Orientations of $K_4 - e$ with a Double Edge

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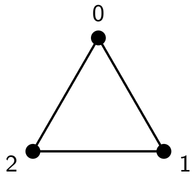
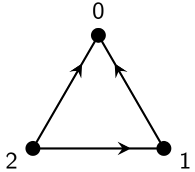
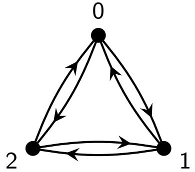
Graphs and Directed Graphs

- Graphs are objects comprising of a vertex set and an edge set, where the edges connect the vertices.
- In a directed graph, or digraph, the edges (called arcs) are assigned directions and are treated as ordered pairs.
- Each different way to assign directions to the edges is called an orientation.
- The complete digraph on n vertices, denoted K_n^*, is the digraph with the arcs (a, b) and (b, a) between every pair of vertices a and b.
- The number of arcs in K_n^* is $n(n-1)$.





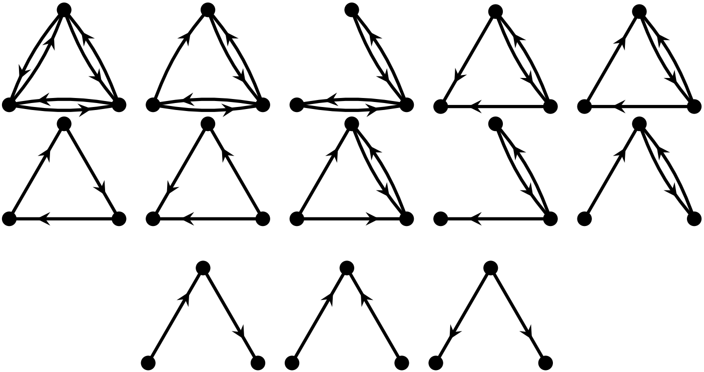


Digraph Decompositions

- If D and H are digraphs with D a subgraph of H, then a D-decomposition of H is a partition of the arc set of H into subgraphs isomorphic to D, called $D-$blocks.
- The spectrum for a digraph D is the set of all n for which a D-decomposition of K_n^* exists.
- Let $V(K_n^*) = \mathbb{Z}_n$ and let D be a subgraph of K_n^*. Define rotating D or clicking D as the isomorphism $i \mapsto i+1$ for each vertex in $V(D)$.
- A D-decomposition of K_n^* is cyclic if clicking D preserves the D-blocks of the decomposition.

Some Background and Previous Results

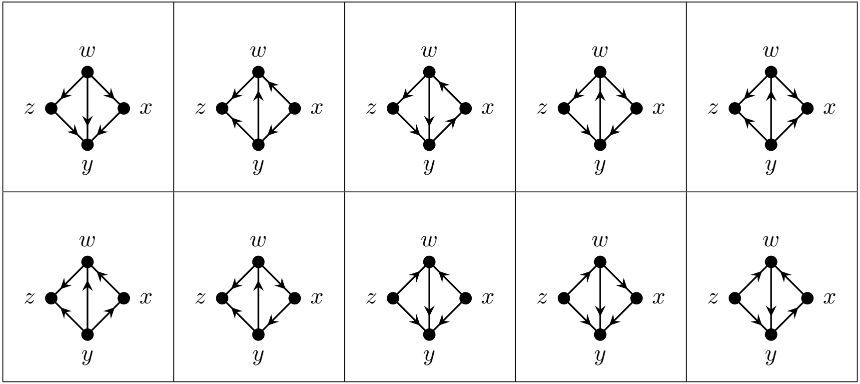
Hartman and Mendelsohn (1986) found the spectra for all subgraphs of K_3^*.[1]



[1] **A. Hartman and E. Mendelsohn**, **The Last of the Triple Systems**, **Ars Combin. 22 (1986)**, **25-41**.

Some Background and Previous Results

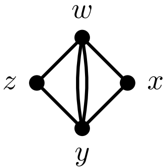
Bunge et al. (2017)– submitted found spectra for almost all of the below 10 orientations of $K_4 - e$.[2]



[2] R. C. Bunge, B. D. Darrow Jr., T. M. Dubczuk, Mentor, Competition Entrant, G. L. Keller, G. A. Newkirk, and D. P. Roberts, On Decomposing the Complete Symmetric Digraph into Orientations of K4-e, Discussiones Mathematicae-Graph Theory, submitted.

Our Research Question

- We want to find the spectrum for all D such that D is an orientation of $K_4 - e$ with a double edge.



- When the graph is oriented, one of the two double edges from w to y must be directed towards w and the other one must be directed towards y.
- The digraphs are named using the conventions in *An Atlas of Graphs* by Read and Wilson.

Some Useful Observations

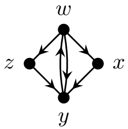
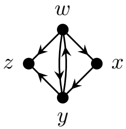
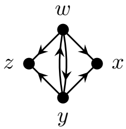
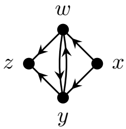
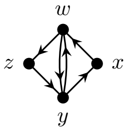
The orientation of a digraph D may be reversed by changing the direction of the arrow on each arc. We denote the reverse orientation as $\mathrm{Rev}(D)$.
Observe that if D and H are digraphs, then a D-decomposition of H exists if and only if a $\mathrm{Rev}(D)$-decomposition of $\mathrm{Rev}(H)$ exists.
Since $K_n^* \cong \mathrm{Rev}(K_n^*)$, we have:

Observation

If D is a digraph, then D decomposes K_n^* if and only if $\mathrm{Rev}(D)$ decomposes K_n^*.

Our Digraphs

After writing out the 16 possible orientations and taking out any digraphs that were isomorphic and/or reverses of each other, we obtained 5 digraphs of interest:

$\mathrm{D125}[w,x,y,z]$	$\mathrm{D110}[w,x,y,z]$	$\mathrm{D106}[w,x,y,z]$	$\mathrm{D118}[w,x,y,z]$	$\mathrm{D135}[w,x,y,z]$
				

Necessary Conditions for Digraph Designs

The following are the necessary conditions for a D-decomposition of K_n^* to exist.

Order condition: $|V(D)| \leq n$.

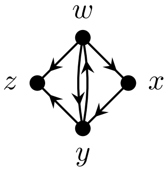
Size condition: $|E(D)|$ divides $n(n-1)$.

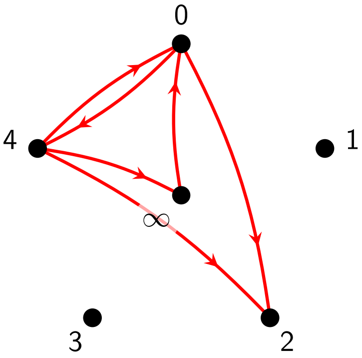
Degree condition: Both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$, which is both the indegree and outdegree of every vertex in K_n^*.

- Need $6|n(n-1)$ by condition 2. Thus $n \equiv 0, 1, 3$, or $4 \mod 6$.
- Need $n \geq 4$ by condition 1.

D110-Decomposition of K_6^*

We now demonstrate some designs for small n which use the aforementioned clicking mechanism.





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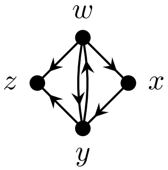

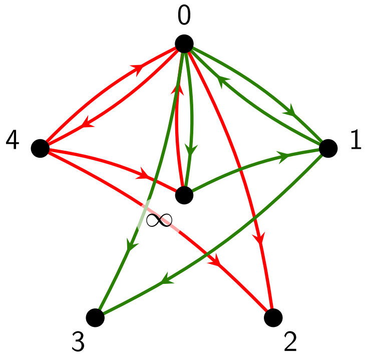

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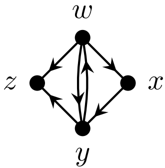

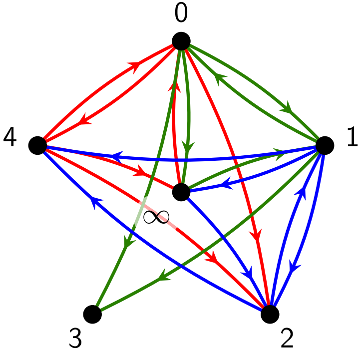

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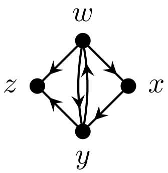

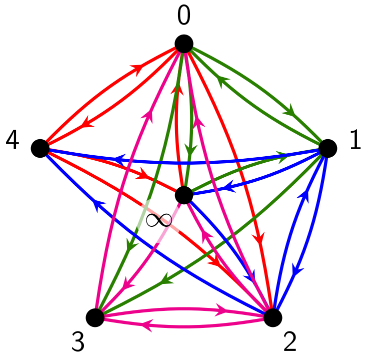

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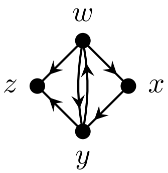
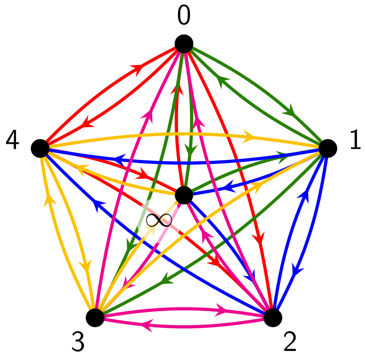
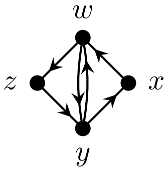
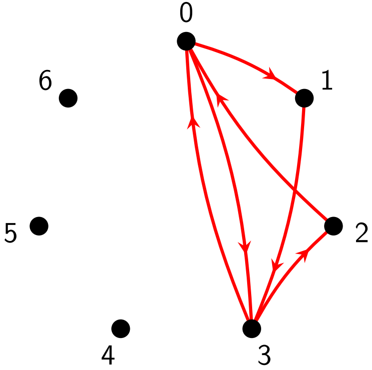




D135-Decomposition of K_7^*





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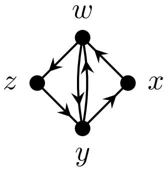

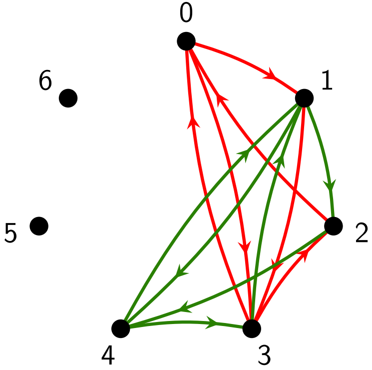

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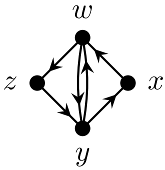

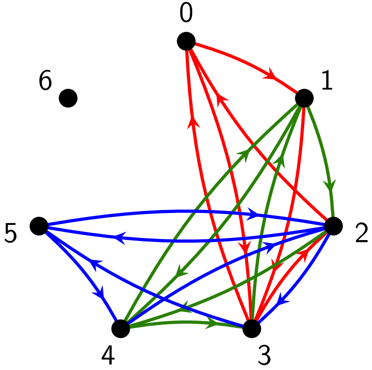

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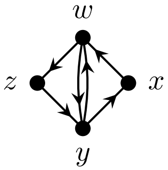

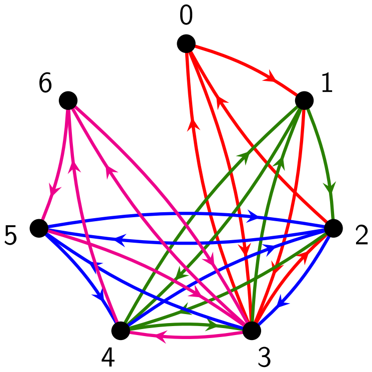

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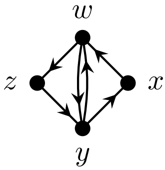

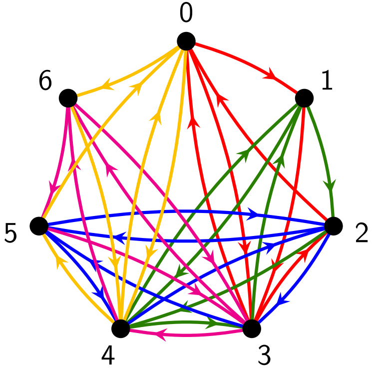

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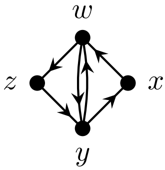

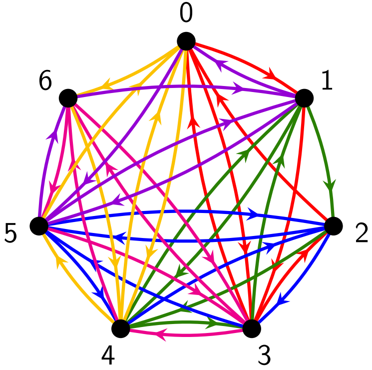

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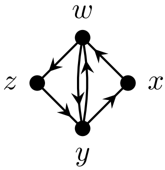
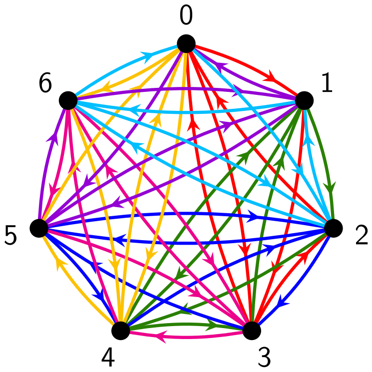
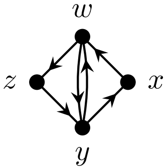




Main Idea

- Not all decompositions need to be cyclic; some can be manual. Finding these manual decompositions was aided by a computer.
- A digraph D may be considered numerically as a set of ordered pairs (Indegree, Outdegree) for each vertex.
- K_6^* may be represented as $\{(5,5),(5,5),(5,5),(5,5),(5,5),(5,5)\}$.
- D135, which is below, may be represented as $[w,x,y,z] \rightarrow \{(2,2),(1,1),(2,2),(1,1)\}$.



Main Idea

- Decompositions can be considered similarly by "adding" the set of ordered pairs corresponding to each $D-$block to create the graph we wish to decompose.
- The set can be "permuted" in any way.
- To decompose K_6^* with $\mathrm{D135}$, start with $\{(0,0),(0,0),(0,0),(0,0),(0,0),(0,0)\}$.
- Permute the set: $w \to 5, x \to 3, y \to 6, z \to 4$.
- Add the permutation to the graph, which then becomes $\{(0,0),(0,0),(1,1),(1,1),(2,2),(2,2)\}$.

Manual $\mathrm{D}135$-Decomposition of K_6^*

- $\{(0,0),(0,0),(0,0),(0,0),(0,0),(0,0)\}$
- $\{(0,0),(0,0),(1,1),(1,1),(2,2),(2,2)\}$
- $\{(2,2),(1,1),(1,1),(1,1),(4,4),(3,3)\}$
- $\{(3,3),(1,1),(3,3),(1,1),(5,5),(5,5)\}$
- $\{(4,4),(3,3),(4,4),(3,3),(5,5),(5,5)\}$
- $\{(5,5),(5,5),(5,5),(5,5),(5,5),(5,5)\}$

Code Algorithm

- The code does the above process backwards: it starts at the end product (i.e. $\{(5,5),(5,5),(5,5),(5,5),(5,5),(5,5)\}$) and subtracts set permutations until we reach $\{(0,0),(0,0),(0,0),(0,0),(0,0),(0,0)\}$.
- The code also uses a memoization technique in order to reduce runtime.
- In the memoization technique, the code stores previous failed runs so it does not have to recompute the same negative result again in the future.
- Many important building blocks were found using the code.

Impossibility Results

- There does not exist a $\mathrm{D}106$- decomposition of K^*_{6p} or K^*_{6q+3} for any $p \geq 1$, $q \geq 1$.
- There does not exist a $\mathrm{D}118$- decomposition of K^*_{6p} or K^*_{6q+4} for any $p \geq 1$, $q \geq 1$.

Impossibility Results–Example Proof

- There does not exist a $\mathrm{D118}$- decomposition of K^*_{6p} or K^*_{6q+4} for any $p \geq 1$, $q \geq 1$.
- Apply necessary condition C, which is:
- Both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$.
- $\gcd\{\text{outdegree}(v) : v \in V(\mathrm{D118})\} = \gcd(2,2,2,0) = 2$. By condition C, if a $\mathrm{D118}$- decomposition of K^*_n existed, then we must have $2|(n-1)$. Thus, $(n-1) \equiv 0 \mod 2$. However, $(6p-1) \equiv (6q+3) \equiv 1 \mod 2 \not\equiv 0 \mod 2$, so the necessary condition fails and thus there does not exist a $\mathrm{D118}$- decomposition of K^*_{6p} or K^*_{6q+4} for any $p \geq 1$, $q \geq 1$.

Blow-up Constructions

- In building general constructions from small cases, we utilize the following theorems from previous literature:[3]
- If n is odd, then a $\{K_3, K_5\}$-decomposition of K_n exists.
- The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{u \times m}$ are
  (i) $u \geq 3$,
  (ii) $(u-1)m \equiv 0 \pmod 2$, and
  (iii) $u(u-1)m^2 \equiv 0 \pmod 6$.
- If $u \geq 3$ and $u \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{u \times 2,4}$.
- Let m, r, s, t, u_1, u_2, ..., u_m all be positive integers. If there exists a $\{K_r, K_s\}$-decomposition of $K_{u_1,u_2,\ldots,u_m}$, then there also exists a $\{K_{r \times t}, K_{s \times t}\}$-decomposition of $K_{tu_1,tu_2,\ldots,tu_m}$.

[3] **C. J. Colbourn and J. H Dinitz (Editors)**, **Handbook of Combinatorial Designs**, **2nd ed.**, **Chapman & Hall/CRC Press**, **Boca Raton**, **FL**, **2007**.

Blow-up Constructions

- If $n \equiv 0 \mod 6$ and $n \geq 6$, then a (K_n^*, D) design exists for $D \in \{\mathrm{D135}\}$.
- If $n \equiv 1 \mod 6$ and $n \geq 7$, then a (K_n^*, D) design exists for $D \in \{\mathrm{D135}\}$.

Conclusions and Future Work

- We were able to make general constructions and impossibility arguments for some of the cases.
- The majority of cases produced partial results.
- Our code could be optimized to reduce runtime and memory usage, as both were impediments when trying to brute-force through larger cases.

Acknowledgements

- We'd like to thank our mentor, Dr. Saad El-Zanati, without whose guidance and support this project would have been impossible.

$\square$

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