

# On a Computer-Aided Decomposition of the Complete Digraph into Orientations of $K_4 - e$ with a Double Edge

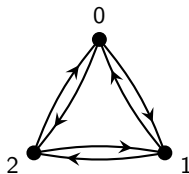
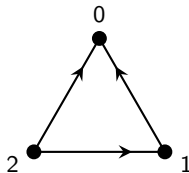
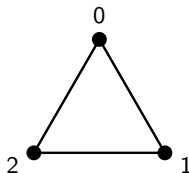
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International Student Science Fair, IMSA, June 30, 2018

## Graphs and Directed Graphs

- **Graphs** are objects comprising of a vertex set and an edge set, where the edges connect the vertices.
- In a **directed graph**, or **digraph**, the edges (called **arcs**) are assigned directions and are treated as ordered pairs.
- Each different way to assign directions to the edges is called an **orientation**.
- The **complete digraph on  $n$  vertices**, denoted  $K_n^*$ , is the digraph with the arcs  $(a, b)$  and  $(b, a)$  between every pair of vertices  $a$  and  $b$ .
- The number of arcs in  $K_n^*$  is  $n(n - 1)$ .

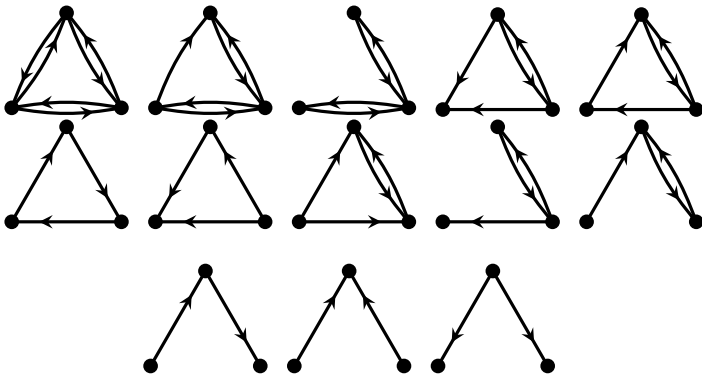


## Digraph Decompositions

- If  $D$  and  $H$  are digraphs with  $D$  a subgraph of  $H$ , then a  **$D$ -decomposition** of  $H$  is a partition of the arc set of  $H$  into subgraphs isomorphic to  $D$ , called  **$D$ -blocks**.
- The **spectrum** for a digraph  $D$  is the set of all  $n$  for which a  $D$ -decomposition of  $K_n^*$  exists.
- Let  $V(K_n^*) = \mathbb{Z}_n$  and let  $D$  be a subgraph of  $K_n^*$ . Define **rotating**  $D$  or **clicking**  $D$  as the isomorphism  $i \mapsto i + 1$  for each vertex in  $V(D)$ .
- A  $D$ -decomposition of  $K_n^*$  is **cyclic** if clicking  $D$  preserves the  $D$ -blocks of the decomposition.

## Some Background and Previous Results

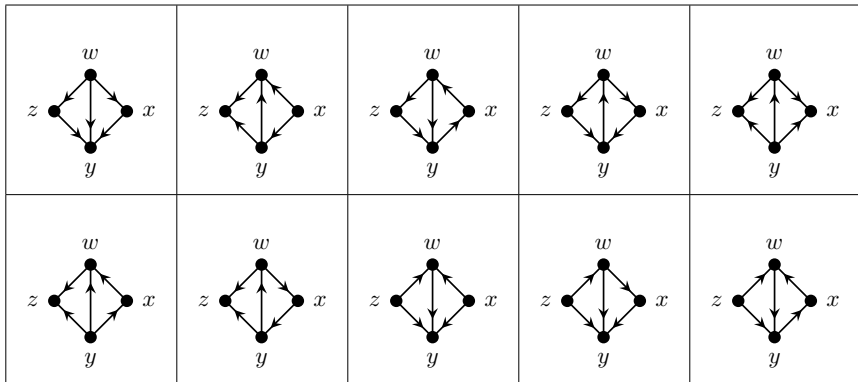
Hartman and Mendelsohn (1986) found the spectra for all subgraphs of  $K_3^*$ .<sup>1</sup>



<sup>1</sup>A. Hartman and E. Mendelsohn, The Last of the Triple Systems, *Ars Combin.* 22 (1986), 25-41.

## Some Background and Previous Results

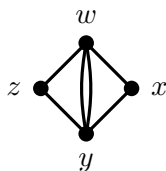
Bunge et al. (2017)– submitted found spectra for almost all of the below 10 orientations of  $K_4 - e$ .<sup>2</sup>



<sup>2</sup>R. C. Bunge, B. D. Darrow Jr., T. M. Dubczuk, Mentor, Competition Entrant, G. L. Keller, G. A. Newkirk, and D. P. Roberts, On Decomposing the Complete Symmetric Digraph into Orientations of  $K_4 - e$ , *Discussiones Mathematicae-Graph Theory*, submitted.

## Our Research Question

- We want to find the spectrum for all  $D$  such that  $D$  is an orientation of  $K_4 - e$  with a double edge.



- When the graph is oriented, one of the two double edges from  $w$  to  $y$  must be directed towards  $w$  and the other one must be directed towards  $y$ .
- The digraphs are named using the conventions in *An Atlas of Graphs* by Read and Wilson.

## Some Useful Observations

The orientation of a digraph  $D$  may be reversed by changing the direction of the arrow on each arc. We denote the reverse orientation as  $\text{Rev}(D)$ .

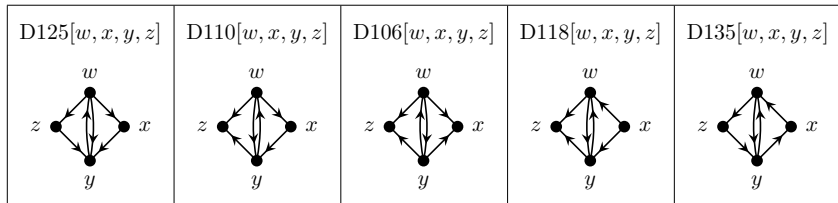
Observe that if  $D$  and  $H$  are digraphs, then a  $D$ -decomposition of  $H$  exists if and only if a  $\text{Rev}(D)$ -decomposition of  $\text{Rev}(H)$  exists. Since  $K_n^* \cong \text{Rev}(K_n^*)$ , we have:

### Observation

*If  $D$  is a digraph, then  $D$  decomposes  $K_n^*$  if and only if  $\text{Rev}(D)$  decomposes  $K_n^*$ .*

## Our Digraphs

After writing out the 16 possible orientations and taking out any digraphs that were isomorphic and/or reverses of each other, we obtained 5 digraphs of interest:





## Necessary Conditions for Digraph Designs

The following are the necessary conditions for a  $D$ -decomposition of  $K_n^*$  to exist.

**Order condition:**  $|V(D)| \leq n$ .

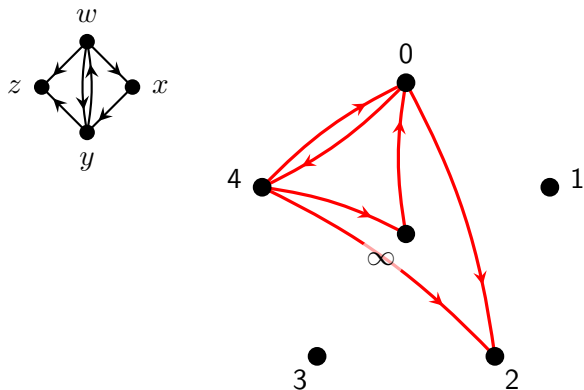
**Size condition:**  $|E(D)|$  divides  $n(n-1)$ .

**Degree condition:** Both  $\gcd\{\text{outdegree}(v) : v \in V(D)\}$  and  $\gcd\{\text{indegree}(v) : v \in V(D)\}$  divide  $n-1$ , which is both the indegree and outdegree of every vertex in  $K_n^*$ .

- Need  $6|n(n-1)$  by condition 2. Thus  $n \equiv 0, 1, 3, \text{ or } 4 \pmod{6}$ .
- Need  $n \geq 4$  by condition 1.

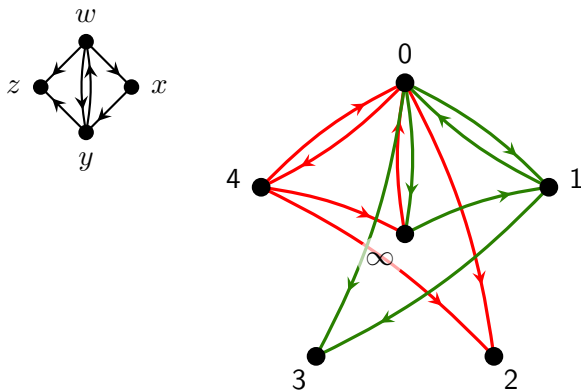
## D110-Decomposition of $K_6^*$

We now demonstrate some designs for small  $n$  which use the aforementioned clicking mechanism.



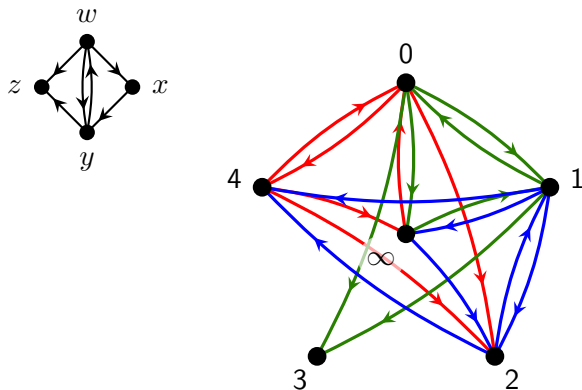
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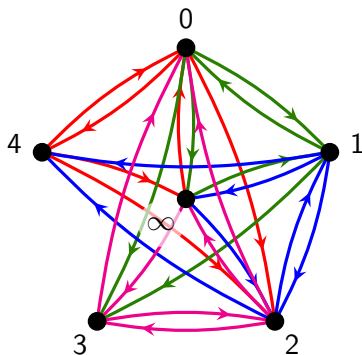
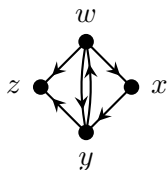
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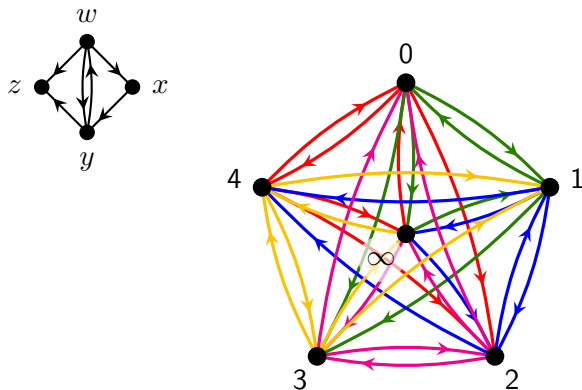
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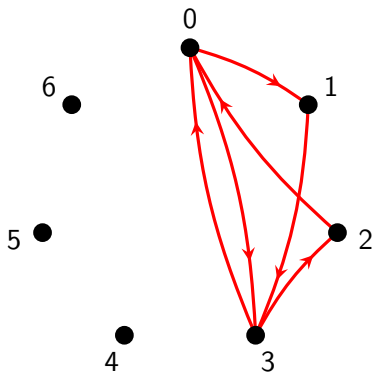
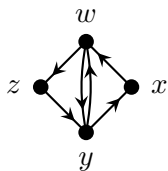


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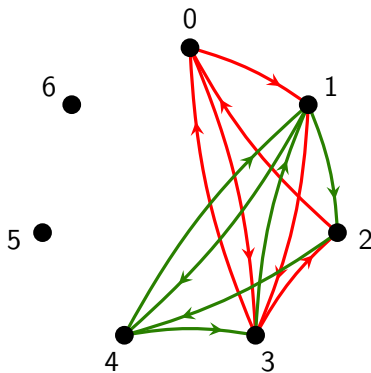
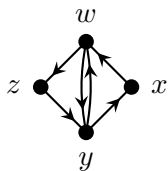
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# D135-Decomposition of $K_7^*$

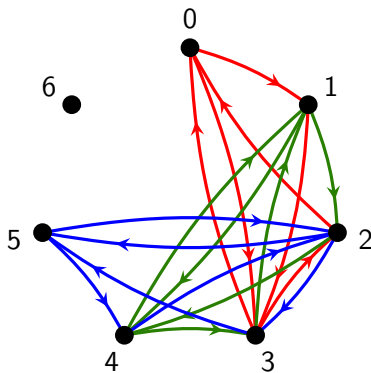
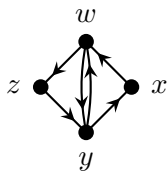


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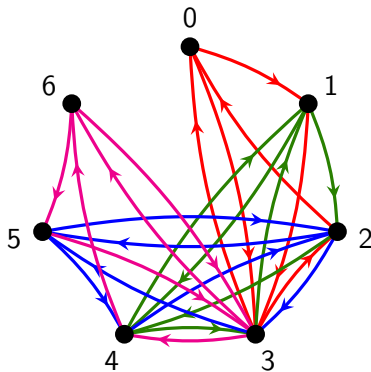
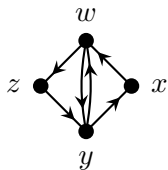




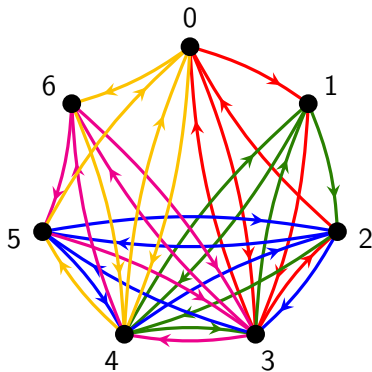
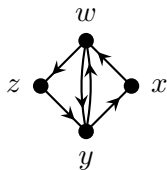
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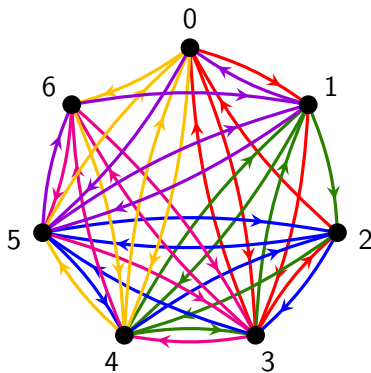
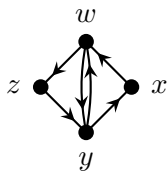
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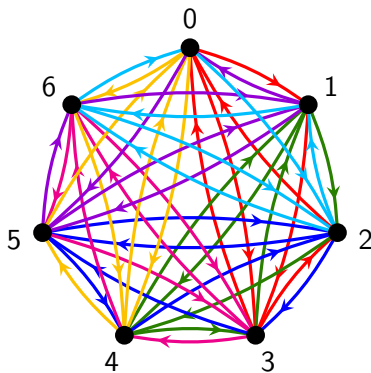
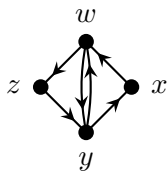
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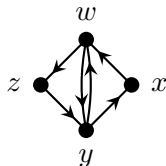


# D135-Decomposition of $K_7^*$



## Main Idea

- Not all decompositions need to be cyclic; some can be **manual**. Finding these manual decompositions was aided by a computer.
- A digraph  $D$  may be considered numerically as a set of ordered pairs (Indegree, Outdegree) for each vertex.
- $K_6^*$  may be represented as  $\{(5, 5), (5, 5), (5, 5), (5, 5), (5, 5), (5, 5)\}$ .
- D135, which is below, may be represented as  $[w, x, y, z] \rightarrow \{(2, 2), (1, 1), (2, 2), (1, 1)\}$ .



## Main Idea

- Decompositions can be considered similarly by "adding" the **set** of ordered pairs corresponding to each  $D$ -block to create the **graph** we wish to decompose.
- The set can be "permuted" in any way.
- To decompose  $K_6^*$  with D135, start with  $\{(0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)\}$ .
- Permute the set:  $w \rightarrow 5, x \rightarrow 3, y \rightarrow 6, z \rightarrow 4$ .
- Add the permutation to the graph, which then becomes  $\{(0, 0), (0, 0), (1, 1), (1, 1), (2, 2), (2, 2)\}$ .

## Manual D135-Decomposition of $K_6^*$

- $\{(0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)\}$
- $\{(0, 0), (0, 0), (1, 1), (1, 1), (2, 2), (2, 2)\}$
- $\{(2, 2), (1, 1), (1, 1), (1, 1), (4, 4), (3, 3)\}$
- $\{(3, 3), (1, 1), (3, 3), (1, 1), (5, 5), (5, 5)\}$
- $\{(4, 4), (3, 3), (4, 4), (3, 3), (5, 5), (5, 5)\}$
- $\{(5, 5), (5, 5), (5, 5), (5, 5), (5, 5), (5, 5)\}$



## Code Algorithm

- The code does the above process backwards: it starts at the end product (i.e.  $\{(5, 5), (5, 5), (5, 5), (5, 5), (5, 5), (5, 5)\}$ ) and subtracts set permutations until we reach  $\{(0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)\}$ .
- The code also uses a memoization technique in order to reduce runtime.
- In the memoization technique, the code stores previous failed runs so it does not have to recompute the same negative result again in the future.
- Many important building blocks were found using the code.

## Impossibility Results

- There does not exist a D106- decomposition of  $K_{6p}^*$  or  $K_{6q+3}^*$  for any  $p \geq 1, q \geq 1$ .
- There does not exist a D118- decomposition of  $K_{6p}^*$  or  $K_{6q+4}^*$  for any  $p \geq 1, q \geq 1$ .

## Impossibility Results—Example Proof

- There does not exist a D118- decomposition of  $K_{6p}^*$  or  $K_{6q+4}^*$  for any  $p \geq 1$ ,  $q \geq 1$ .
- Apply necessary condition C, which is:
- Both  $\gcd\{\text{outdegree}(v) : v \in V(D)\}$  and  $\gcd\{\text{indegree}(v) : v \in V(D)\}$  divide  $n - 1$ .
- $\gcd\{\text{outdegree}(v) : v \in V(\text{D118})\} = \gcd(2, 2, 2, 0) = 2$ . By condition C, if a D118- decomposition of  $K_n^*$  existed, then we must have  $2|(n - 1)$ . Thus,  $(n - 1) \equiv 0 \pmod{2}$ . However,  $(6p - 1) \equiv (6q + 3) \equiv 1 \pmod{2} \not\equiv 0 \pmod{2}$ , so the necessary condition fails and thus there does not exist a D118- decomposition of  $K_{6p}^*$  or  $K_{6q+4}^*$  for any  $p \geq 1$ ,  $q \geq 1$ .

## Blow-up Constructions

- In building general constructions from small cases, we utilize the following theorems from previous literature:<sup>3</sup>
- If  $n$  is odd, then a  $\{K_3, K_5\}$ -decomposition of  $K_n$  exists.
- The necessary and sufficient conditions for the existence of a  $K_3$ -decomposition of  $K_{u \times m}$  are
  - (i)  $u \geq 3$ ,
  - (ii)  $(u - 1)m \equiv 0 \pmod{2}$ , and
  - (iii)  $u(u - 1)m^2 \equiv 0 \pmod{6}$ .
- If  $u \geq 3$  and  $u \equiv 0 \pmod{3}$ , then there exists a  $K_3$ -decomposition of  $K_{u \times 2, 4}$ .
- Let  $m, r, s, t, u_1, u_2, \dots, u_m$  all be positive integers. If there exists a  $\{K_r, K_s\}$ -decomposition of  $K_{u_1, u_2, \dots, u_m}$ , then there also exists a  $\{K_{r \times t}, K_{s \times t}\}$ -decomposition of  $K_{tu_1, tu_2, \dots, tu_m}$ .

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<sup>3</sup>C. J. Colbourn and J. H. Dinitz (Editors), *Handbook of Combinatorial Designs*, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

## Blow-up Constructions

- If  $n \equiv 0 \pmod{6}$  and  $n \geq 6$ , then a  $(K_n^*, D)$  design exists for  $D \in \{D135\}$ .
- If  $n \equiv 1 \pmod{6}$  and  $n \geq 7$ , then a  $(K_n^*, D)$  design exists for  $D \in \{D135\}$ .

## Conclusions and Future Work

- We were able to make general constructions and impossibility arguments for some of the cases.
- The majority of cases produced partial results.
- Our code could be optimized to reduce runtime and memory usage, as both were impediments when trying to brute-force through larger cases.

# Acknowledgements

- We'd like to thank our mentor, Dr. Saad El-Zanati, without whose guidance and support this project would have been impossible.





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