The Entangling Properties of Knots and Links

Comparing Quantum Entanglement and Topological Entanglement

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Outline of Presentation

1. Major Results
   - Knot Invariants
2. Definitions and Concepts
   - Knots and Links
   - Knot Polynomials
   - Entanglement
3. The Hypothesis
   - Q.L.I's
4. Theorem and Proof
   - Theorem

Outline of Presentation

Knot Invariants Need Quantum Entanglement

Theorem

Non-entangling R-matrices cannot form topological invariants

- Entanglement: acting on a particle at one place will influence it very, very far away.
- Non-entangling: a physical process (operator) that cannot form entangled states from non-entangled states
- Knot Invariants: A unique property of a knot.
- This entire investigation is motivated by a pun!

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Knots and Links

- Knots and links ⇔ rubber sheet geometry (topology)
- A knot acts just like a closed loop of rope.

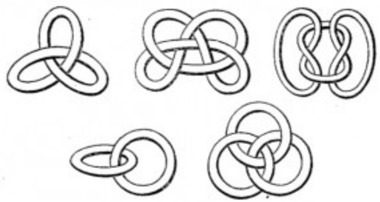


Figure: Examples of Knots and Links

Outline of Presentation

Using Polynomials to Distinguish Knots

- Each knot has a unique Jones polynomial.
- The Jones polynomial has many connections to quantum physics.
- The Jones polynomial is related to the Kauffman bracket polynomial - the knot invariant we used in this study.

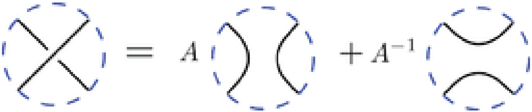


Figure: The Bracket Relation

Reidemeister Moves and Topology

- Satisfying these moves will preserve topological properties

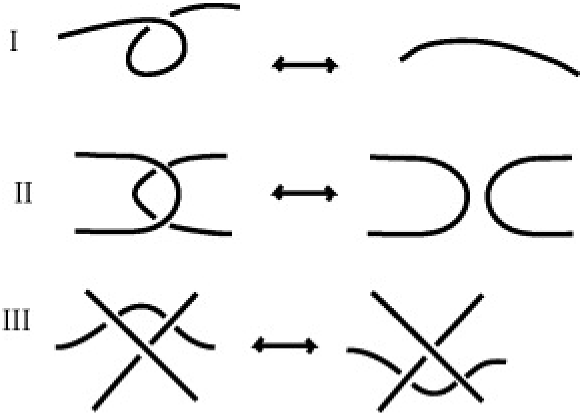
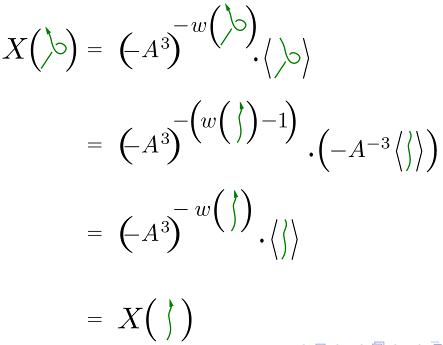


Figure: The Reidemeister moves

The Bracket Polynomial

- The bracket satisfies Reidemeister moves II and III.
- It does NOT satisfy Reidemeister I.
- We can introduce a corrective factor to account for Reidemeister I.



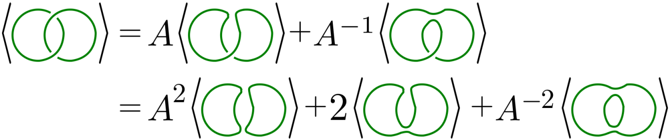
$$X(\ ) = (-A^3)^{-w(\ )} \cdot \langle\ \rangle$$

$$= (-A^3)^{-(w(\ )-1)} \cdot (-A^{-3}\langle\ \rangle)$$

$$= (-A^3)^{-w(\ )} \cdot \langle\ \rangle$$

$$= X(\ )$$

Example Calculation of the Bracket



$$= (A^2 + A^{-2}) \cdot (-A^2 - A^{-2}) + 2$$

$$= -A^4 - A^{-4}$$

Figure: Bracket of the Hopf link

General and Normalized Bracket Forms

- The general form of the bracket is given by

$$< K >= \sum_{S} < K|S > d^{|S|-1}$$

, where $d = (-A^2 - A^{-2})$.

- The normalization of the oriented bracket is given by

$$f_k = (-A^3)^{-w(K)} < K > \in \mathbb{Z}[A, A^{-1}]$$

, where w(K) is the writhe (sum of oriented crossings).

Outline of Presentation

Quantum Mechanics

- Quantum Mechanics is a probabilistic theory of nature.
- Quantum processes are transformations on a complex vector space.

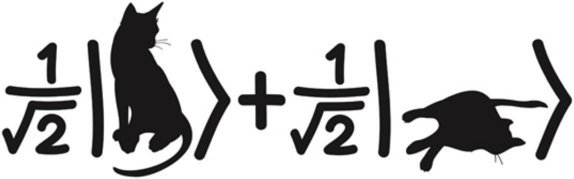


Figure: Quantum Mechanics

Entanglement

- Entanglement is the correlation of quantum states.
- Mathematically, we say that a quantum process G is entangling if there is a vector

$$|\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \in V \otimes V$$

such that $G\,|\alpha\beta\rangle$ cannot be written as a tensor product.

Remark

We have methods to measure how entangling an operator is.

Outline of Presentation

Quantum Link Invariants

- Knots can be represented by particles moving through space with respect to time.
- These particles are created in pairs (cups) and annihalate subsequently in pairs (caps).

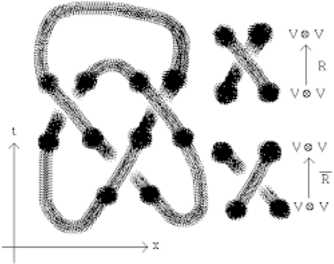


Figure: A knot formed from vacuum-vacuum processes

- Diagrammatically, we can break up a knot into pieces.
- Each piece can be given a "quantum" matrix representation.

$$R = Overcrossing \tag{1}$$

$$\bar{R} = Undercrossing \tag{2}$$

$$M = Cup\ or\ Cap \tag{3}$$

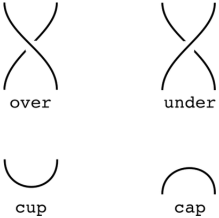


Figure: Parts of a knot

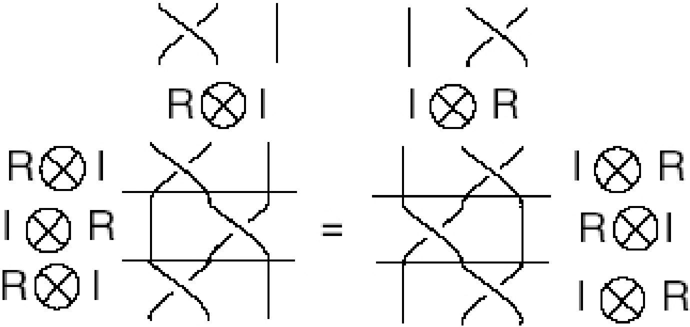


Figure: Quantum link invariants in action!

- Cups/caps M and crossings R in knots can be given matrix representations.

$$R = \begin{pmatrix} (A^{-1}) & 0 & 0 & 0 \\ 0 & (-A^2 + A^{-1}) & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & (A^{-1}) \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & (iA) \\ (-iA^{-1}) & 0 \end{pmatrix}$$

Statement of the Problem

- From an *R* matrix like the one shown, topological invariants can be produced.
- All the examples we can calculate show that entanglement is a necessary condition.

Conjecture

Can topology and entanglement be linked through this process?

Outline of Presentation

Theorem

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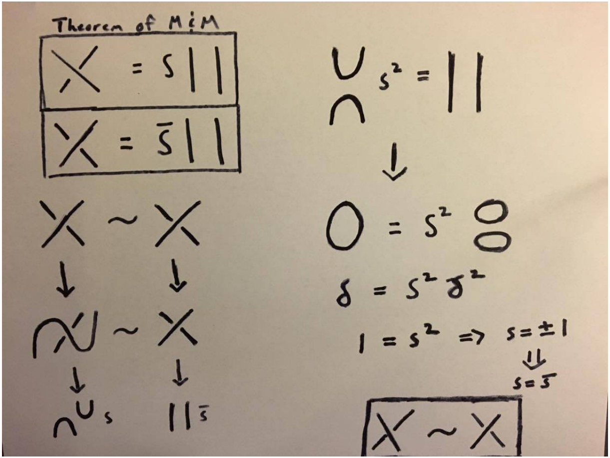
Non-entangling R matrices cannot be used to produce topological invariants of quantum links

Proof (Special Case)



Summary

- Quantum link invariants need entanglement in order to be topological invariants.
- Topology, Knot Theory, and Quantum Theory are intimately related.
- We still do not understand the extent of the relationship between entanglement and topology.
- Outlook
  - We may be able to employ some techniques from gauge theory and algebraic topology to better understand entanglement strength.

Further Reading

- Kauffman, L. Lomonaco, S. (2010). Topological quantum information theory. *Journal of the American Mathematical Society*, *68* (1), 103-176.
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- Witten, E. (1989). Quantum field theory and the jones polynomial. *Communications in Mathematical Physics*, *121*, 351-399.