The Mathematical Wonders of Pascal's Triangle

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1. In the expansion of $(x+y)^5$, what is the coefficient of each of the following terms? (And I'm thinking now would be a great time to put a TI-89 or Wolfram Alpha to use!)

a. x^4y b. x^3y^2 c. x^2y^3

2. In the expansion of $(x+y)^7$, what is the coefficient of each of the following terms?

a. x^5y^2 b. x^3y^4 c. y^7

3. In the expansion of $(x+y)^{10}$, what is the coefficient of each of the following terms?

a. x^7y^3 b. x^5y^5 c. x^4y^6

4. In the expansion of $(x+y)^{12}$, what is the coefficient of each of the following terms?

a. x^8y^4 b. x^5y^7 c. x^2y^{10}

5. What is the value of each of the following?

a. $\binom{5}{1}$ b. $\binom{5}{2}$ c. $\binom{5}{3}$

d. $\binom{7}{2}$ e. $\binom{7}{4}$ f. $\binom{7}{7}$

g. $\binom{10}{3}$ h. $\binom{10}{5}$ i. $\binom{10}{6}$

j. $\binom{12}{4}$ k. $\binom{12}{7}$ l. $\binom{12}{10}$

6. Based on the results of problems #1-5 (plus any other examples you wish to try), what is the coefficient of each of the following terms in the expansion of $(x+y)^n$, where n is a positive integers greater than 3, written in the form $\binom{n}{r}$.

a. $x^{n-1}y^1$ b. x^2y^{n-2} c. $x^{n-3}y^3$ d. y^n

7. **<u>Conjecture</u>** – In the expansion of $(x+y)^n$, if n and r are positive integers with $n \geq r$, then what is the coefficient of the $x^{n-r}y^r$ term?

8. In how many ways can 7 distinct trading cards be divided between John and Diana if John and Diana receive the following number of trading cards, respectively?

a. 6 cards, 1 card b. 4 cards, 3 cards c. 2 cards, 5 cards

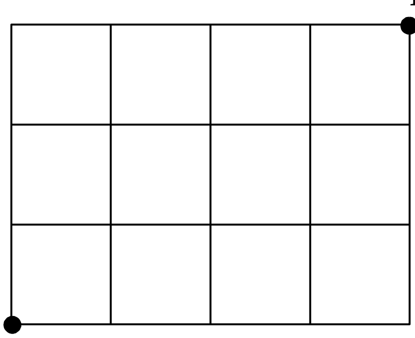
9. In how many ways can 10 distinct trading cards be divided between John and Diana if John and Diana receive the following number of trading cards, respectively?

a. 8 cards, 2 cards b. 6 cards, 4 cards c. 3 cards, 7 cards

10. In the figures below, how many different paths can be taken from point A to point B if you may only travel right or up?

a.

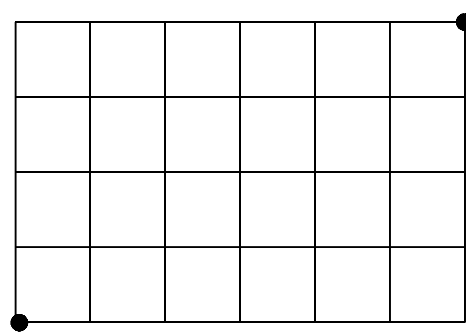
B



A

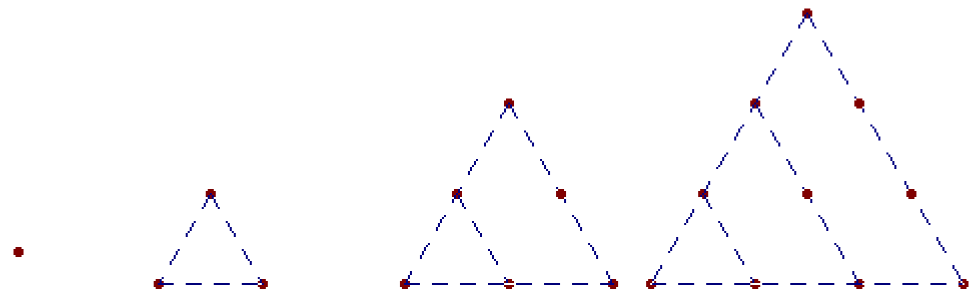
b.

B

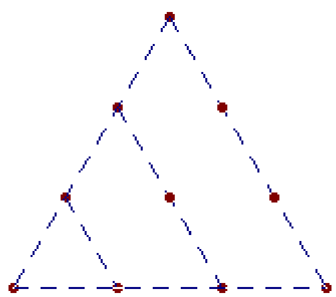


A

Below are drawings of the first four Triangular Numbers. (Can you guess why they are called Triangular Numbers?)



In the space below, use the fourth Triangular Number to help you draw the fifth Triangular Number.



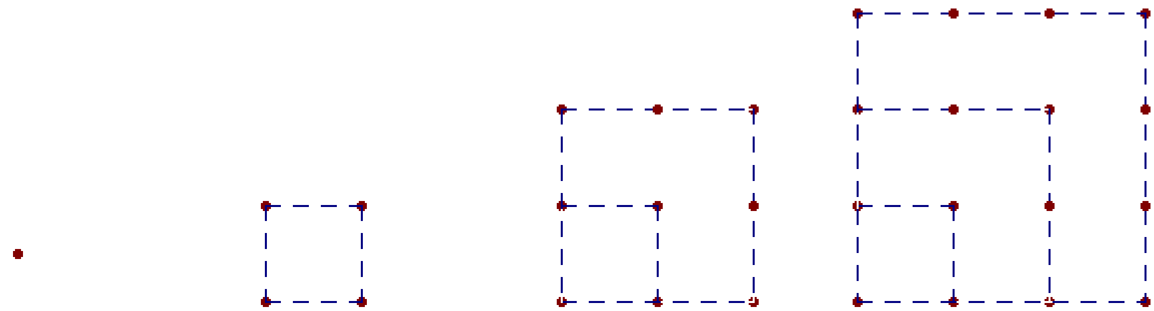
Now, see if you can complete the table below.

Triangular Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	3	6	10				…	

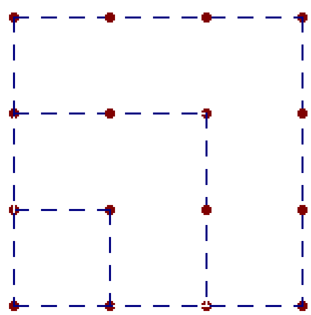
If you haven't figured out a method for determining the *n*th Triangular Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ________________________________

Below are drawings of the first four Square Numbers. (Can you guess why they are called Square Numbers?)



In the space below, use the fourth Square Number to help you draw the fifth Square Number.

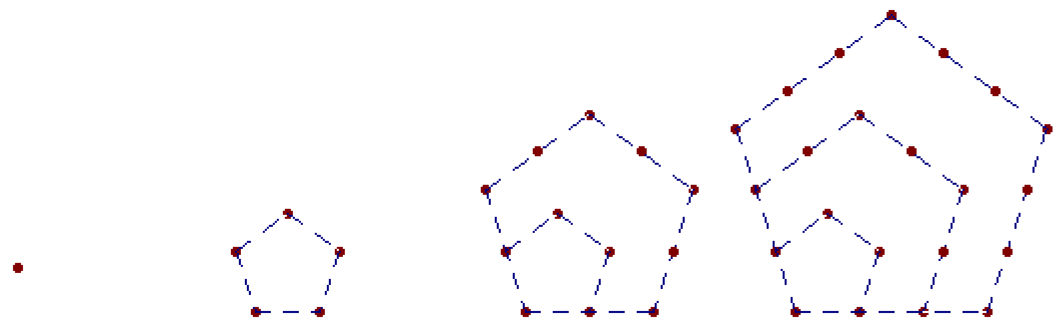


Now, see if you can complete the table below.

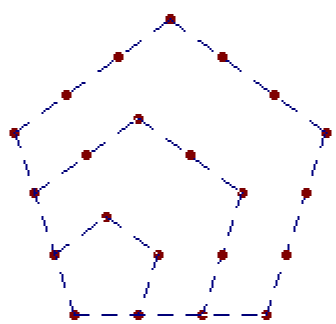
Square Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	4	9	16				…	

If you haven't figured out a method for determining the *n*th Square Number, consider using the process of finite differences to help you discover it.

Below are drawings of the first four Pentagonal Numbers. (Can you guess why they are called Pentagonal Numbers?)



In the space below, use the fourth Pentagonal Number to help you draw the fifth Pentagonal Number.



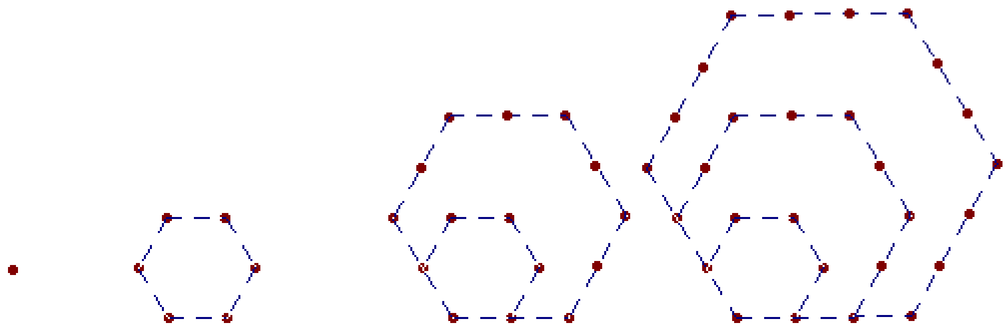
Now, see if you can complete the table below.

Pentagonal Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	5	12	22				…	

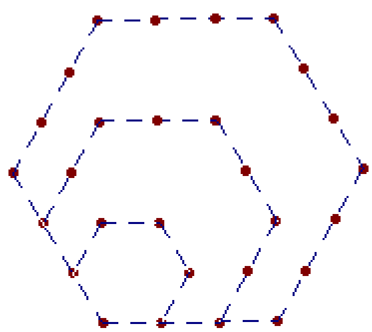
If you haven't figured out a method for determining the *n*th Pentagonal Number, consider using the process of finite differences to help you discover it.

Polygonal Numbers Worksheet Name: ______________________________

Below are drawings of the first four Hexagonal Numbers. (Can you guess why they are called Hexagonal Numbers?)



In the space below, use the fourth Hexagonal Number to help you draw the fifth Hexagonal Number.

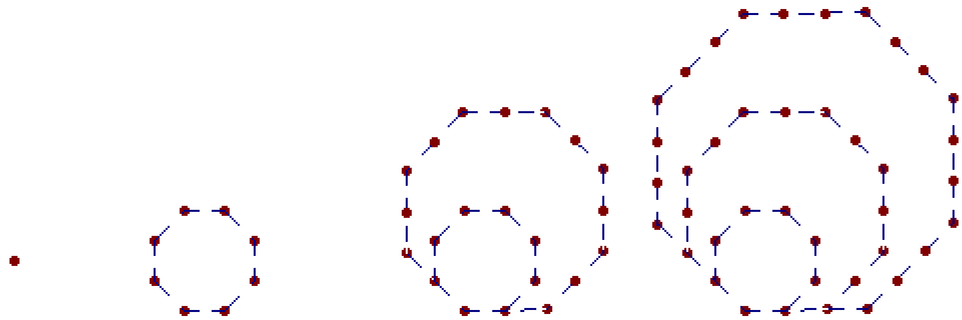


Now, see if you can complete the table below.

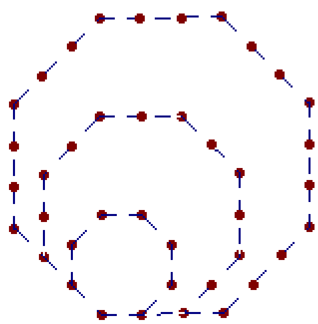
Hexagonal Numbers									
	1st	2nd	3rd	4th	5th	6th	7th	…	*n*th
Number of dots	1	6	15	28				…	

If you haven't figured out a method for determining the *n*th Hexagonal Number, consider using the process of finite differences to help you discover it.

Below are drawings of the first four Octagonal Numbers. (Can you guess why they are called Octagonal Numbers?)



In the space below, use the fourth Octagonal Number to help you draw the fifth Octagonal Number.



Now, see if you can complete the table below.

Octagonal Numbers										
	1st	2nd	3rd	4th	5th	6th	7th	8th	…	*n*th
Number of dots	1	8	21	40					…	

If you haven't figured out a method for determining the *n*th Octagonal Number, consider using the process of finite differences to help you discover it.

1	**2**	**3**	**4**	**5**	**6**	**7**	**…**	***n***
1	1						…	
1	2						…	
1	3	6	10	15	21	28	…	$\frac{n(n+1)}{2}$
1	4	9	16	25	36	49	…	n^2
1	5	12	22	35	51	70	…	$\frac{n(3n-1)}{2}$
1	6	15	28	45	66	91	…	$n(2n-1)$
1	7						…	
1	8						…	
1	9							
1	10							

1. In the table above, fill in the third through seventh polygonal numbers for the:
   a. heptagonal numbers (7th row)
   b. octagonal numbers (8th row)
   c. nonagonal numbers (9th row)
   d. decagonal numbers (10th row)
   e. “two”-gonal (2nd row)
   e. “one”-gonal numbers (1st row)

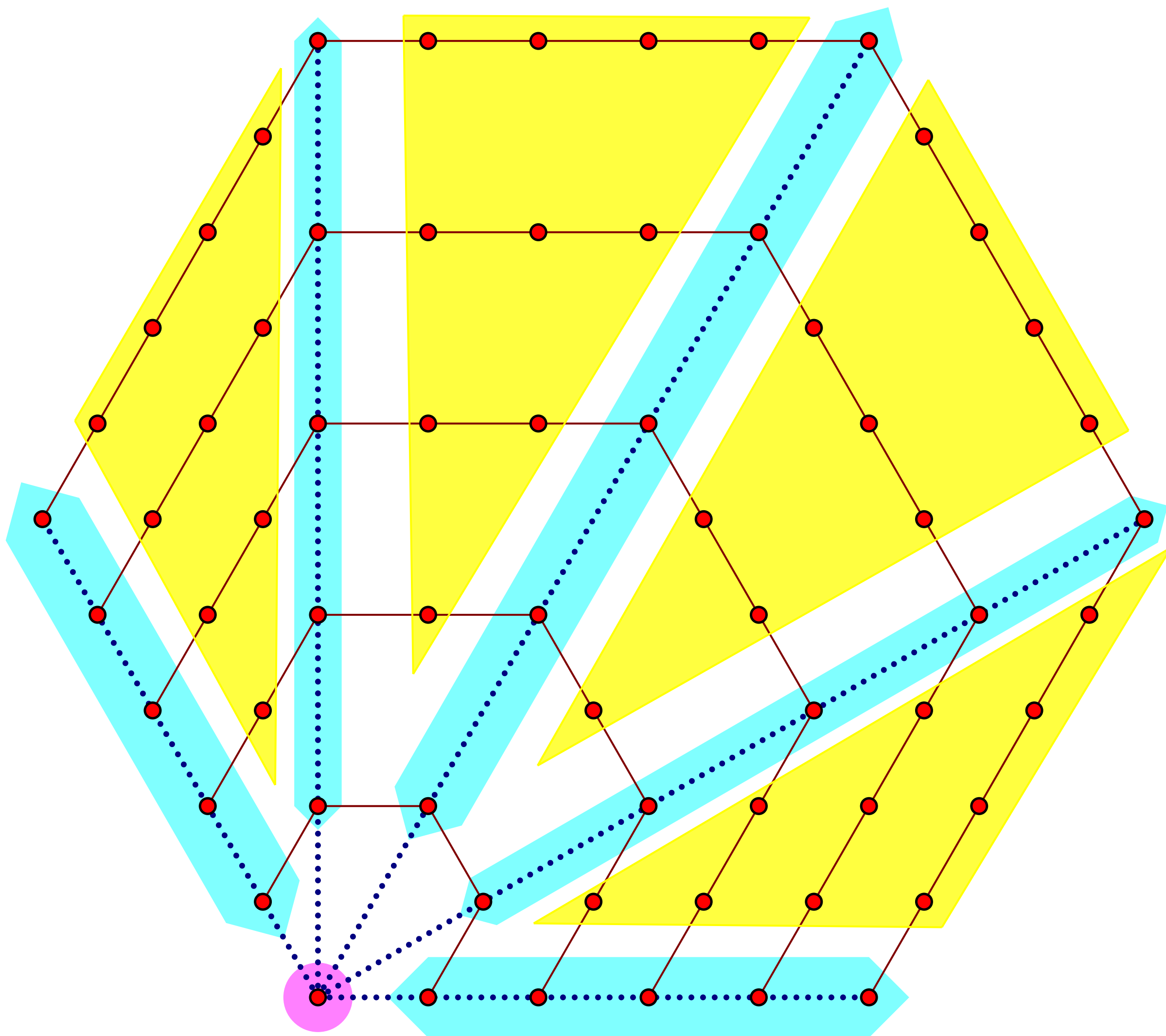
2. Find each general polygonal number and enter it in the appropriate position in the table above. Recall that P(*m*, *n*) refers to the *n*th “*m*”-gonal number so that P(3, *n*) = $\frac{n(n+1)}{2}$, and P(4, *n*) = n^2.
   a. P(7, *n*) b. P(8, *n*) c. P(9, *n*) d. P(10, *n*) e. P(2, *n*) f. P(1, *n*)

3. Use the formulas for the *n*th "*m*"-gonal numbers in the table above to derive a formula for P(*m*, *n*), the general *n*th "*m*"-gonal number.

4. Look for (at least) one pattern that relates entries from different rows of polygonal number (like the relationship between triangular and square numbers we discussed). In the space below describe this relationship (and any others you find).

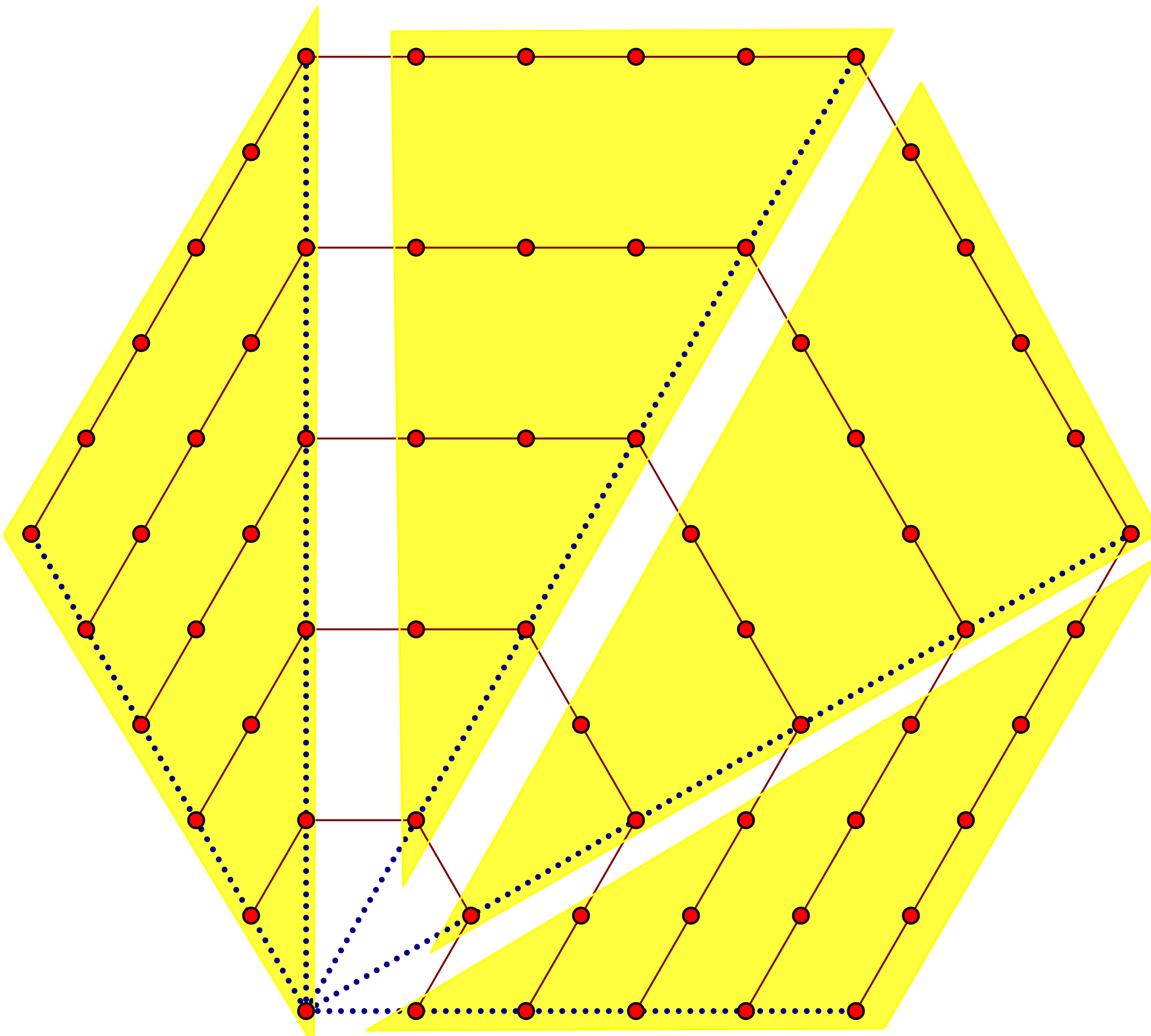
The m^{th} n-gonal number is

$$1+(n-1)(m-1)+\frac{(n-2)(n-1)}{2}\cdot(m-2)$$

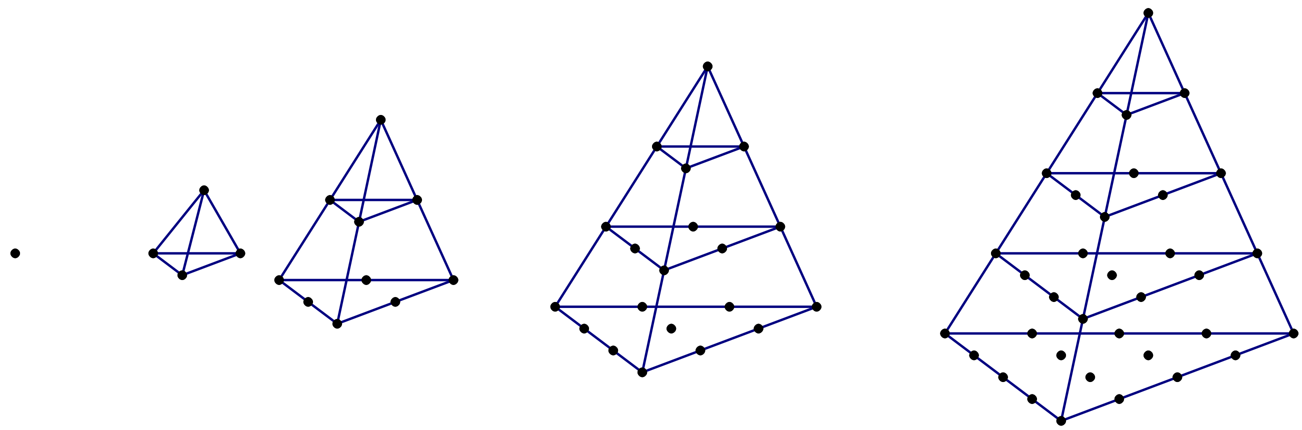


The m^{th} n-gonal number is

$$\frac{n(n+1)}{2}+\frac{n(n-1)}{2}\cdot(m-3)$$

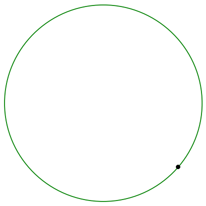
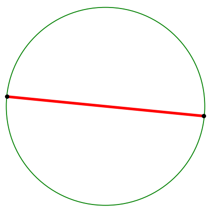
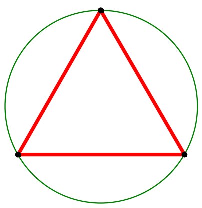
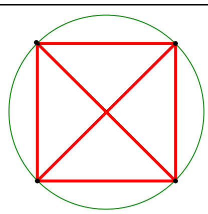
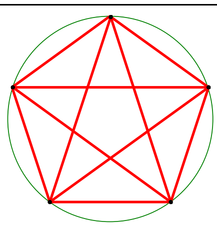
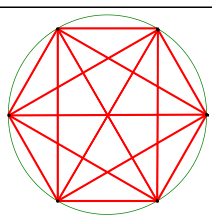
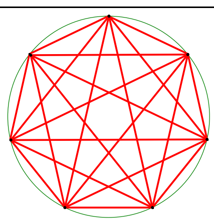


Tetrahedral Numbers



	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
Tetrahedral	1	4	10	20	35				

Points on a Circle Worksheet Name: ______________________________

	Points	Segments	Triangles	Quads	Pentagons	Hexagons	Heptagons
							
							
							
							
							
							
							

The "Hats" Problem

A group of *n* men enter a restaurant and check their hats. The hat checker is absent minded, and when they leave, she redistributes the hats back to the men at random. What is the probability P_n that no man gets his own hat, and how does P_n behave as *n* approaches infinity?

Pierre de Montmort, 1713,
Essay d' Analyse sur les Jeus de Hazard

# of Hats and People	Number of Matches of Hats to People					
	0	**1**	**2**	**3**	**4**	**5**
0						
1						
2						
3						
4						
5						

Define H(*n*, *r*) to be the number of ways of distributing *n* hats to *n* people so that exactly *r* people receive the correct hat. Fill in the table above (and on the following page) with appropriate values of H(*n*, *r*). Don't forget to use patterns in the values to help you fill in all the entries in the table.

Name: ______________________________

N of Matches of Hats to People	Number of Hats and People			
	6	**7**	**8**	**9**
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				

Solving the Hat Matching Problem

Recall that for notational purposes, H(n, r) is defined to be the number of ways of distributing n hats to n people so that exactly r people receive the correct hat.

# of Hats and People	Number of Matches of Hats to People									
	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	1	0	1							
3	2	3	0	1						
4	9	8	6	0	1					
5	44	45	20	10	0	1				
6	265	264	135	40	15	0	1			
7	1854	1855	924	315	70	21	0	1		
8	14833	14832	7420	2464	630	112	28	0	1	
9	133496	133497	66744	22260	5544	1134	168	36	0	1

Key Relationship

$H(n,m) = H(n-m,0)\cdot\binom{n}{m}$ or $H(n,n-k) = H(k,0)\cdot\binom{n}{n-k}$

Explanation

Consider $H(n, m)$, which is the number of ways to distribute n hats to n people so that exactly m people get the correct hat. This can be done by first choosing the m people out of n who will get the correct hat. This can be done in $\binom{n}{m}$ ways.

Now, for each single instance of m people getting their correct hat, you must distribute the remaining $n - m$ hats to the $n - m$ people so that none gets the correct hat. This can be done in $H(n - m, 0)$ ways. Thus, the total number of ways to distribute n hats to n people so that exactly m people get the correct hat is $H(n,m) = H(n-m,0)\cdot\binom{n}{m}$.

Room to Room Worksheet Name: ______________________________

How many different ways can you travel from Room 1 to each of the following room in the figure below if, when you leave a room, you can only travel to an adjacent room that has a higher room number than the room you just left?

1. Room 7
2. Room 10
3. Room 13