NOTES

Logistics: This lesson is intended for students in Grades 6 - 8 in an introductory pre-algebra or geometry course as a hands-on investigation into several geometry concepts. This specific lesson utilizes manipulatives, called **tangrams**, to explore area, ratios and isometric transformations in the coordinate plane.

Materials:

per student:

1 – copy of Student Pages

per pair of students:

2 – Sets of tangram manipulatives ("tans") numbered appropriately (see Advanced Preparation)
4 – TANSformation Grid Papers on 11"x17" paper
1 – Large manila folder or other object to serve as a privacy divider
2 – Angle Rulers
2 – Miras
2 – Pieces of uncooked linguine
2 – Rulers
Writing Utensils
Colored Pencils (Optional)
Computer with Internet Access (Optional)

Time: Two to three 50-minute class periods

Objectives/Standards:

- Use problem-solving skills and the application of basic characteristics of two-dimensional shapes to design mathematical puzzles. **CCSS.Math.Practice.MP1, CCSS.Math.Practice.MP2, CCSS.Math.Practice.MP5, CCSS.Math.Practice.MP6**
- Describe a ratio relationship between the areas of geometric figures, including triangles and quadrilaterals, and apply this understanding to mathematical puzzles consisting of composite shapes. **CCSS.Math.Content.6.RP.A.1, CCSS.Math.Content.6.G.A.1, CCSS.Math.Content.7.G.B.6, CCSS.Math.Practice.MP2, CCSS.Math.Practice.MP3, CCSS.Math.Practice.MP7**
- Using translations, rotations and reflections, design a mathematical puzzle on the coordinate plane consisting of two-dimensional figures. **CCSS.Math.Content.8.G.A.3, CCSS.Math.Practice.MP2, CCSS.Math.Practice.MP3, CCSS.Math.Practice.MP4, CCSS.Math.Practice.MP6**

Tangram Transformations

NOTES

Introduction: One of the oldest and most recognizable series of mathematical puzzles is the **tangram**. Although the complete origin of these geometric puzzles is inconsistent, it is said that they were first published in 1813 in Imperial China, and finally reached the United States and Europe around 1817. This dissection puzzle, consisting of 7 polygons (5 right triangles, 1 square and 1 parallelogram), allows students to apply knowledge of basic two-dimensional shapes, problem solving skills, and pattern recognition to seamlessly piece together these "tans" into a given silhouette. In the following activity, students will explore the geometric characteristics of each **tan** prior to designing their own mathematical puzzle through the application of transformations on the coordinate plane.

Mathematically proficient students understand various ways in which characteristics of basic, two-dimensional shapes can be used to solve geometric puzzles. Additionally, they recognize that a series of rigid, or **isometric transformations**, map a figure from one location to another while preserving shape and congruency. This lesson will allow students to investigate these practices. First, students will identify multiple relationships between the tans; specifically, the area ratio of select pieces. Then, students will have the opportunity to explore a tangram puzzle in preparing for the subsequent activity.

In the second activity, students will use their knowledge of isometric transformations on the coordinate grid to design their own tangram puzzles. During this time, students will identify the sequence of reflections, translations and rotations that map each tan from its original position, or **pre-image**, to its end location, or **image**, within the puzzle. Then, students will take turns reading their puzzle transformations to a partner, who will actively manipulate their own tan pieces based on these directions. Students will then verify that the intended transformations result in the correct mathematical puzzle.

Prior to the start of this activity, students should have an understanding of the characteristics of two-dimensional shapes, area, and ratios. Additionally, it is recommended that they are proficient at locating points and performing isometric transformations on two-dimensional shapes in the coordinate plane. To complete *Activity 2: TANSformations*, students should also be capable of rotating figures about a point (not necessarily the origin) and reflecting figures about a line (not necessarily the x- or y-axis).

Tangram Transformations

Advanced Preparation: Prior to beginning the activity with students, prepare the following materials:

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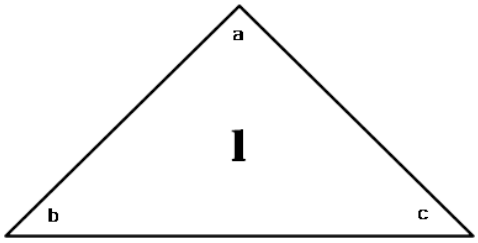
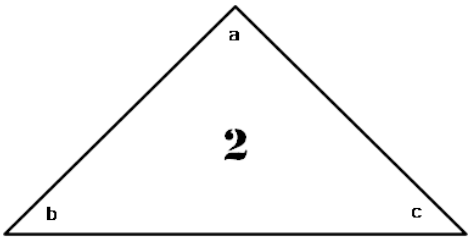
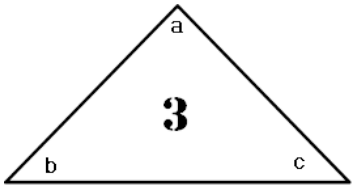
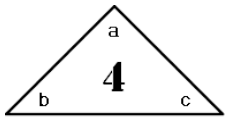
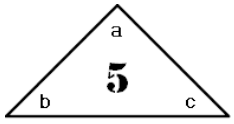
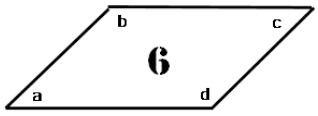
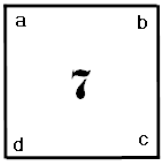
- **Tangram Manipulatives** – Verify that all sets of manipulatives contain 7 polygons. Using a black permanent marker, label the pieces accordingly:

Large Triangle	
Large Triangle	
Medium Triangle	
Small Triangle	
Small Triangle	
Parallelogram	
Square	

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Activity:

Students should be grouped into partner teams for the entirety of this lesson. It is important that students have the opportunity to discuss the investigation and their findings with a peer, as this will lead to a deeper understanding of the mathematical concepts investigated throughout the activities. Additionally, students should periodically pause during the activity to discuss the debriefing and reflection questions as a whole class. This provides another opportunity for mathematical discourse.

Part 1: The Tans of Tangrams

To begin, pass out the first student page titled, **Investigating the TANS**. Explain to the students that they will be investigating geometric manipulatives, called "tans", which are used to solve tangram puzzles. Choose one volunteer to read the opening paragraph aloud.

Provide each partner team with the appropriate materials. Allow several minutes for the students to explore the tans and record their observations. During this time, you may choose to circulate the room to evaluate student discourse and assess their level of prior knowledge.

Once partner teams have completed the initial task, ask student volunteers to share their observations. During this discussion, students may notice:

- **The set includes 7 polygons, including triangles, a parallelogram, and a square.**
- **There are three sizes of right triangles.**
- **Select pieces can be placed together to make other pieces:**
  - *2 small triangles are equal to the medium triangle*
  - *2 small triangles are equal to the parallelogram*
  - *2 small triangles are equal to the square*
  - *2 small triangles and the square are equal to the large triangle (also, 4 small triangles)*
  - *2 small triangles and the parallelogram are equal to the large triangle*
  - *2 small triangles and the medium triangle are equal to the large triangle*
  - *2 small triangles and the square are equal to the large triangle*

Note: In this part of the activity, students should disregard the labels on the tangram pieces.

Tangram Transformations

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Students will now complete the procedures as listed in their student pages. First, they will determine the area of each tan. Provide students with a ruler and colored pencils, if desired. There are a variety of strategies that students may use in order to calculate the areas. For example, they may only find the area of the small right triangle, and then use the shape relationships that were previously identified to determine the area of the remaining shapes. Alternatively, they may outline each polygon on the provided grid paper and use the appropriate formulas, or count the units.

Then, students will compare the areas of two tans in ratio form. Students may choose to write their ratios in fraction form, using the word "to", or with a semi-colon. Regardless of the notation, all ratios should be written in simplest form.

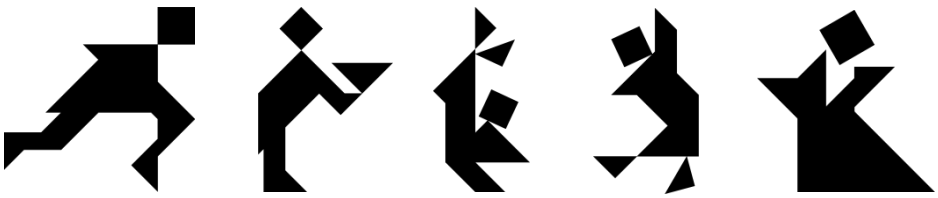
Answer Key:

TAN	AREA
Small Triangle	6.25 cm^2
Medium Triangle	12.5 cm^2
Large Triangle	25 cm^2
Parallelogram	12.5 cm^2
Square	12.5 cm^2

Ratio of the areas of the…	Ratio
Medium Triangle to Small Triangle	2:1
Small Triangle to Large Triangle	1:4
Small Triangle to Parallelogram	1:2
Large Triangle to Parallelogram	2:1
Medium Triangle to Square	1:1
Large Triangle to Square	2:1
Medium Triangle to Parallelogram	1:1

***Note:** Provided answers are calculated using the ETA hands2minds Overhead Tangrams. These can be located at:*

http://www.hand2mind.com/item/overhead-tangrams-set-of-4/5478



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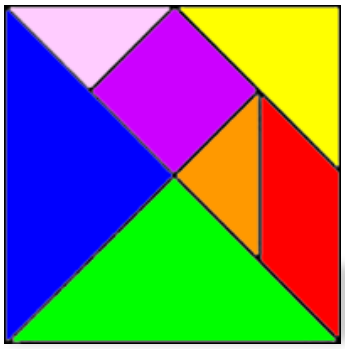
Once each partner team has completed their work, reconvene as a whole class. Pose the following questions:

- **What method did you use to determine the area of each tan?** [*Students could measure the side lengths and height of each polygon and calculate the area using appropriate area formulas, trace each figure on the provided grid paper and count individual units, or use their understanding of the mathematical relationships between each shape to determine the area.*]
- **Were there any discrepancies in the areas you calculated depending on the method you used? Explain.** [*All areas should be approximately the same regardless of what method was used. Slight differences may occur due to estimation and inaccurate measurements.]*
- **Explain the relationship between the pieces, and use your understanding of ratios to support your justifications.** [*The small triangle can be used to construct all other shapes. Using this logic, it can also be used to determine the area of all tan pieces.]*

While students present the answers to these questions, encourage other partner teams to offer additional information and justify, or refute, their peers' responses. This review process allows the students to review prior knowledge, investigate multiple ways to solve a problem, and presents students with the opportunity to make connections between their understanding of geometry concepts and that of their peers.

In the final part of this activity, students will attempt their first tangram puzzle. Using their seven tans, students are challenged to individually build a **SQUARE,** and are then asked to sketch their solution. Upon completion of the Tangram puzzle, each student should compare their puzzle with that of their partner. Following a brief discussion about the similarities and differences between their puzzles, students will complete the reflection questions listed in their student pages.



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Debrief Part 1:

In a whole class discussion, address the following questions:

- ***Describe the strategy that you used to solve the tangram puzzle.*** [*Answers will vary. Some students will simply guess and check the location of each piece while others will use their understanding of the basic characteristics and areas of the tan pieces to determine their location.]*
- ***Are there multiple ways to complete the puzzle? Explain.*** [*There is technically only one solution. However, because this figure has multiple lines of symmetry, pieces may be reflected or rotated within the figure.]*
- ***How could understanding the areas of each tan help you solve a tangram puzzle?*** [*Students may use their understanding of side lengths and angles to determine where they may be placed within the tangram puzzle. Additionally, they may recognize that the larger triangles must be equal to half of the square's total area and use this understanding to strategically place these tan pieces.]*

OPTIONAL: At this time, students may wish to practice additional tangrams. There are a variety of websites with interactive Tangram puzzles:

- http://pbskids.org/cyberchase/math-games/tanagram-game/
- http://www.abcya.com/tangrams.htm
- http://www.museumofplay.org/web-games/tangrams/

Additionally, this may be an appropriate time for the students to investigate **dilations**. This non-isometric transformation produces an image that is similar to the original. Dilations do not preserve congruency. Students can explore this concept by studying the side lengths of the right triangles. Using their tangram manipulatives, learners may verify that the medium triangle is a dilation of the small triangle by a scale factor of approximately 1.4, or more precisely, $\sqrt{2}$. The large triangle is also a dilation of the medium triangle by a scale factor of approximately 1.4, or more precisely, $\sqrt{2}$.

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Part II: TANSformations

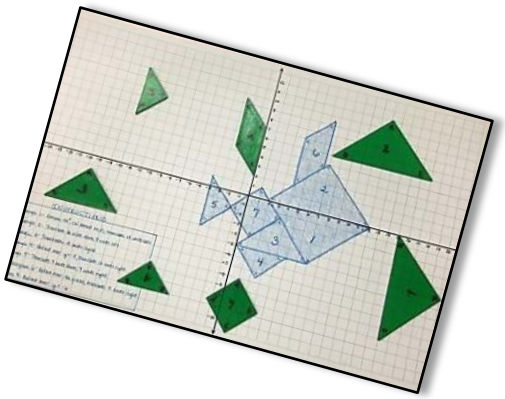
Pass out the next three student pages, the first of which is titled, **TANSformations**. In this portion of the activity, students will combine their understanding of isometric transformations within the coordinate plane and their new knowledge of tangram puzzles. Students will continue to work in partner teams for this portion of the lesson.

To begin, discuss the following questions in a whole class setting. Elicit answers from student volunteers:

- **How do we describe the location of a point on the coordinate plane?** [*As an ordered pair (x,y)]*
- **What are transformations on the coordinate plane?** [*The change of position or direction a point or collection of points on the coordinate plane.]*
- **What happens to a shape when it is translated?** [*The shape "slides" a fixed distance in a certain direction.]*
- **What happens to a shape when it is rotated?** [*The shape "turns" a number of degrees about a fixed point in a given direction.]*
- **What happens to a shape when it is reflected?** [*The shape "flips" over a given line.]*
- **What happens to a shape when it is dilated?** [*The shape enlarges or constricts based on a given scale factor and center of dilation.]*
- **What is a pre-image? What is an image?** [*These points identify the starting location and ending location of a transformed point or figure.]*

Then, select a student to read the Background Information and Challenge statement. Students will continue to work in the same partner teams as in the previous activity.

Explain to the students that they will create their own tangram puzzle using the tan pieces. Then, separated by a privacy divider, students will take turns solving their partner's puzzle by simply listening to a sequence of transformations. Allow several minutes for the students to read through the procedures prior to passing out the necessary materials.



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Teacher Note: Student procedures include the instruction, "Each transformation must be used to form your puzzle." As the instructor, you may choose to differentiate this activity in a variety of ways. For example, you may require all students to use **double transformations** to move their tan pieces in the coordinate plane. Determine the level of complexity at the start of the activity and clearly communicate your expectations to the students. See *Modifications and Extensions* at the end of this lesson for more information.

Provide each partner team with their tangram manipulatives, TANSformation grid paper, measurement tools and privacy divider. Students will then begin designing their own tangram puzzle by using a series of isometric transformations to move each tan from its starting location to a new location within the puzzle. It is important that students record the movement of each tan in the appropriate table of their student pages. Additionally, students should work independently. **It is important that partners do not see one another's puzzles.**

When both partners have completed their puzzles, they will place a privacy divider between their work stations. Each student will then take a second piece of TANSformation grid paper and set up the tans in their original locations. Taking turns, partners will read the series of transformations that form their puzzle. While one partner is reading their directions, the other partner will physically move the tans on their grid paper. Once all tans have been moved, students can then remove the privacy divider to see if they have successfully completed their partner's tangram puzzle.

<u>Debrief Part II:</u>

- ***When developing your own puzzle, how did you decide where each tan would fit?*** [*Student answers will vary. Elicit ideas from multiple volunteers.]*
- ***What was the most difficult part of forming your tangram puzzle?*** [*Student answers will vary. Elicit ideas from multiple volunteers.]*
- ***How do transformations affect the size, shape and location of a figure?*** [*The transformations used in this activity only affect the location of the shape. Translations, rotations and reflections preserve the size and shape of the figures.]*
- ***Two categories of transformations are*** **isometric** ***and*** **non-isometric transformations*****. In this activity, you worked with*** **isometric transformations*****. What do you think is the difference between these two categories of transformations?*** [*Isometric transformations*

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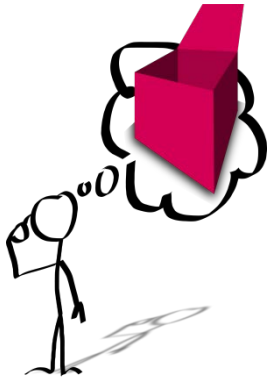
preserve the size and shape of the figures. Non-isometric transformations preserve the shape of the figure, but not the size. Translations, rotations and reflections are examples of isometric transformations while dilations are non-isometric.]

- ***One transformation, dilation, was not addressed in your tangram puzzle. However, several tans could be considered "dilations" of one another if they were correctly mapped on the coordinate plane. Explain.*** [*If mapped onto the coordinate plane, students could investigate that the right triangles could be considered dilations of one another. Using their tangram manipulatives, learners may verify that the medium triangle is a dilation of the small triangle by a scale factor of approximately 1.4, or more precisely,* $\sqrt{2}$*. The large triangle is also a dilation of the medium triangle by a scale factor of approximately 1.4, or more precisely,* $\sqrt{2}$*.*

EXTENSION:

To extend this lesson, students could continue their investigation of isometric and non-isometric transformations on the coordinate plane. Incorporating dilations, students could explore how to enlarge or shrink a two-dimensional figure on the coordinate plane. This could then lead to the investigation of scale factor through the use of the small, medium and large triangles.



Additionally, this activity could be adjusted to meet the needs of various academic levels:

- For lower academic levels, students could simply identify the isometric transformations that would complete their tangram puzzle. By working "backwards", students could first build their tangram puzzle and then determine what transformations would map each tan back to its original location. Students could then read these "inverse" transformations to their partner.

- For more advanced geometry students, learners could identify the composition of transformations using algebriac notation. Additional information on this extension can be found at: http://mathbitsnotebook.com/Geometry/Transformations/TRCompositeTransformations.html

CONCLUSION:

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This lesson is intended to provide students with an opportunity to investigate multiple geometric concepts through the use of manipulatives. Designed to promote student creativity and engagement, learners apply their understanding of isometric transformations to design their own tangram puzzles. This activity also encourages student collaboration as learners work in partner teams to solve and analyze one another's puzzles. For lessons that follow, it is appropriate for students to continue their investigation of isometric and non-isometric transformations. Additionally, based on their academic level, they may begin exploring the symbols and notations that represent a composition of transformations.

RESOURCES:

Composition of Transformations (Isometries) - MathBitsNotebook(Geo - CCSS Math). (n.d.). Retrieved February 03, 2016, from http://mathbitsnotebook.com/Geometry/Transformations/TRCompositeTransformations.html

Dilations. (n.d.). Retrieved February 04, 2016, from http://www.regentsprep.org/regents/math/geometry/gt3/ldilate2.htm

Reviewing Transformations. (n.d.). Retrieved February 04, 2016, from http://www.regentsprep.org/regents/math/algtrig/atp9/reviewTranformations.htm

Investigating the TANS

Tangrams utilize a series of 7 polygons, called "tans". These pieces have specific characteristics, and their basic geometric properties are used to make interesting, and often very challenging, puzzles. Prior to designing your own tangram puzzles, you will investigate these properties using shape recognition, problem solving, and your understanding of mathematical relationships.

Problem: *What mathematical relationships exist between various tans, and how can they be arranged to seamlessly form problem puzzles?*

Materials per group:

- 2 – Sets of tangram manipulatives ("tans")
- 1 – Ruler
- Writing Utensils
- Colored Pencils (Optional)
- Computer with Internet Access (Optional)

Procedure:

1.) Your teacher will begin by passing out two sets of tangram manipulatives to you and your partner. Take several minutes to observe these pieces. Record any observations below:

2.) Now, calculate the area of each tan. You may choose to use a ruler, the provided grid paper, or another mathematical method to complete your measurements. Record your calculations in the area below:

TAN	AREA
Small Triangle	
Medium Triangle	
Large Triangle	
Parallelogram	
Square	

3.) One way to represent the areas of various tans is in ratio form. A **ratio** is a numerical comparison of two items. Complete the following ratios by using the information previously collected. Write all solutions in simplest form.

Ratio of the areas of the…	Ratio
Medium Triangle to Small Triangle	
Small Triangle to Large Triangle	
Small Triangle to Parallelogram	
Large Triangle to Parallelogram	
Medium Triangle to Square	
Large Triangle to Square	
Medium Triangle to Parallelogram	

4.) With your understanding of basic geometric shapes and new knowledge of the tans, complete your first tangram puzzle.

The solution to your tangram puzzle must meet the following requirements:

- **All 7 pieces must be used.**
- **All pieces must lay flat.**
- **All pieces must touch.**
- **No pieces can overlap.**

Arrange your tangram pieces to form a **SQUARE**. Once you have completed the puzzle, sketch an illustration of the square and the location of each tan below:

5.) Complete the following reflection questions with your partner. Be prepared to share your answers with the class:

- Describe the strategy that you used to solve the tangram puzzle.

- Are there multiple ways to complete the puzzle? Explain.

- How does understanding the areas of each tan, and the relationships among the pieces, help you solve a tangram puzzle?

Transformation Tangrams

The Tans of Tangrams Grid Paper

TANSformations

One unique way to study tangrams is to complete these mathematical puzzles on the coordinate plane. In this activity, you will work both individually, and with a partner, to study the various motions of the tan pieces from one location to another. In geometry, these motions are called transformations. Rotations, reflections, translations and dilations are all transformations that occur in the coordinate plane. You will apply your knowledge of transformations by creating your own tangram puzzles.

Challenge: *Using translations, rotations and reflections, design a mathematical puzzle on the coordinate plane using two-dimensional figures.*

Materials per group:

- 2 – Sets of tangram manipulatives ("tans")
- 4 – TANSformation Grid Paper on 11"x17" paper
- 1 – Large manila folder or other object to serve as a privacy divider
- 2 – Angle Rulers, Miras and Pieces of uncooked linguine
- 2 – Rulers
- Writing Utensils and Colored Pencils (Optional)

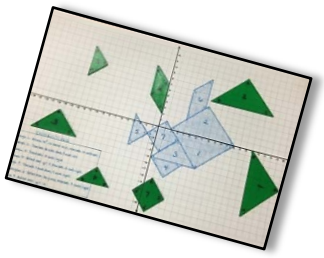
Procedure:

1.) Obtain two pieces of the TANSformation Grid Paper from your instructor. You may choose to label the numeric intervals of each axis.

2.) Place each tan on its starting location on the TANSformation Grid Paper.

3.) Using one piece of the grid paper, create a tangram by using rotations, reflections, translations and/or double transformations to move each tan from its starting position, or **pre-image**, to another location on the coordinate plane. Utilize the appropriate tools to assist in your design. Each transformation must be used at least once to form your puzzle. **Work independently. Do not let your partner see your tangram.**

4.) Using a writing utensil, draw the position of each tan within your puzzle. Label all vertices correctly. The new location of the tan is called the **image.**



Consider the following. Tan pieces…

- may translate in series of one half or one unit.
- may reflect over a line other than the x- and y-axis.
- may rotate over a point other than the origin.
- must follow the rules of traditional tangram puzzles.

5.) Identify the sequence of transformations that you used for each tan:

Shape	Transformation(s)
Large Triangle #1	
Large Triangle #2	
Medium Triangle #3	
Small Triangle #4	
Small Triangle #5	
Parallelogram #6	
Square #7	

6.) When you and your partner have completed your tan puzzle, place the privacy divider between your work areas.

7.) Taking turns, you and your partner will complete each other's puzzles.

- Decide which partner will read their transformation sequences first.
- Using a new piece of grid paper, the other person will begin with each tan on its starting location.
- As one partner reads their transformation sequences, their partner will move each tan accordingly.
- Once all tans have been moved, remove the privacy divider to see if you have successfully completed your partner's tangram puzzle.

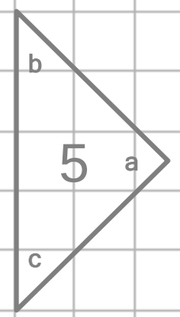
Reflect and Discuss:

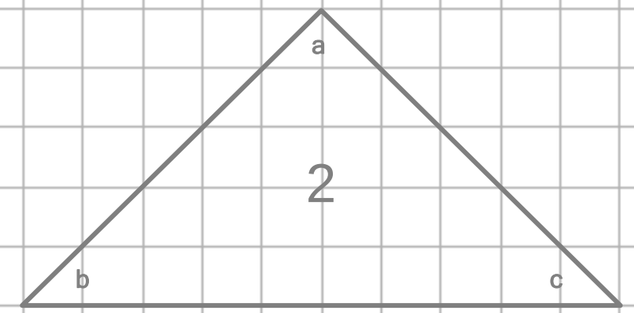
- When developing your own puzzle, how did you decide where each tan would fit?

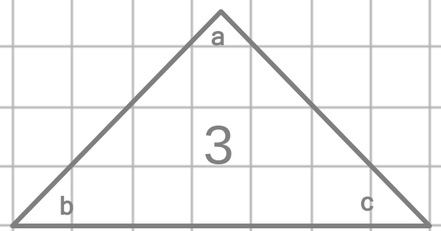
- How do transformations affect the size, shape and location of a figure?

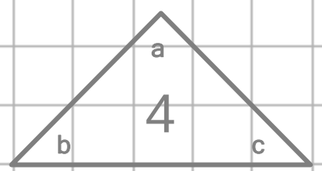
- Two categories of transformations are *isometric* and *non-isometric transformations*. In this activity, you worked with *isometric transformations*. What do you think is the difference between these two categories of transformations?

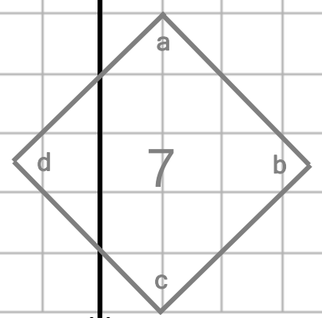
- One transformation, dilation, was not addressed in your tangram puzzle. However, several tans could be considered "dilations" of one another if they were correctly mapped on the coordinate plane. Explain.

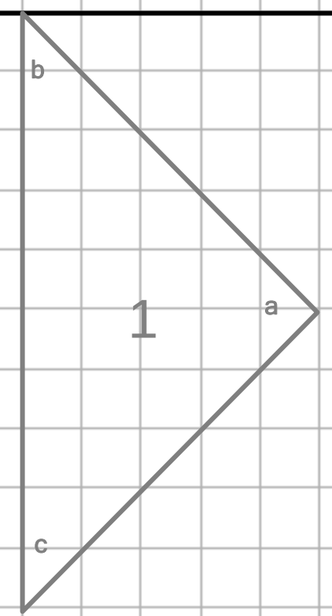


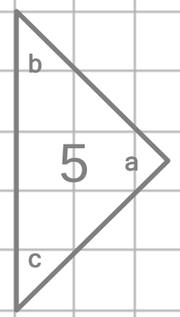


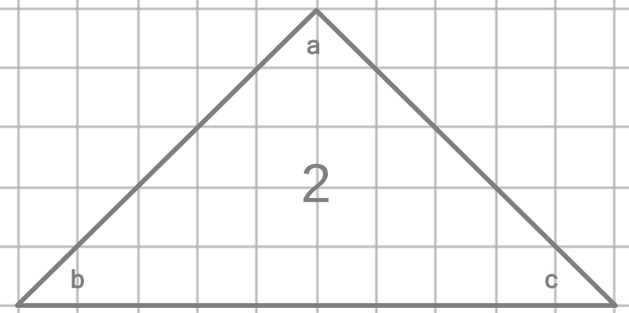


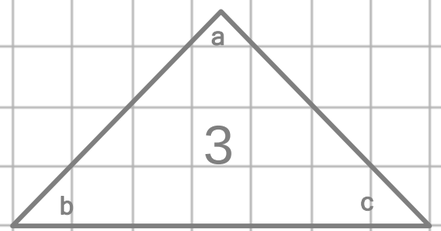


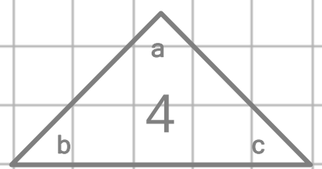


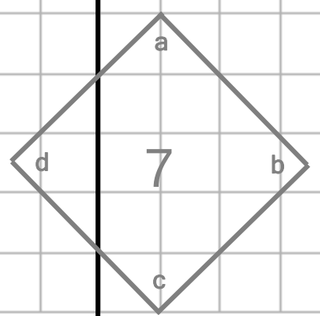


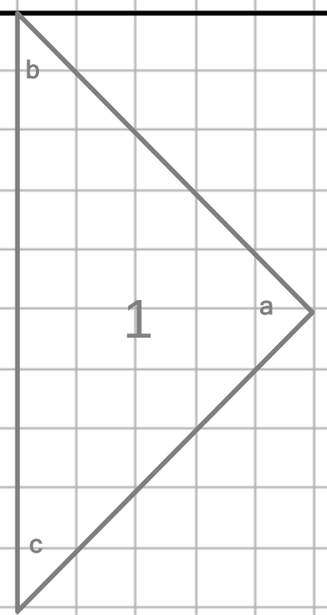


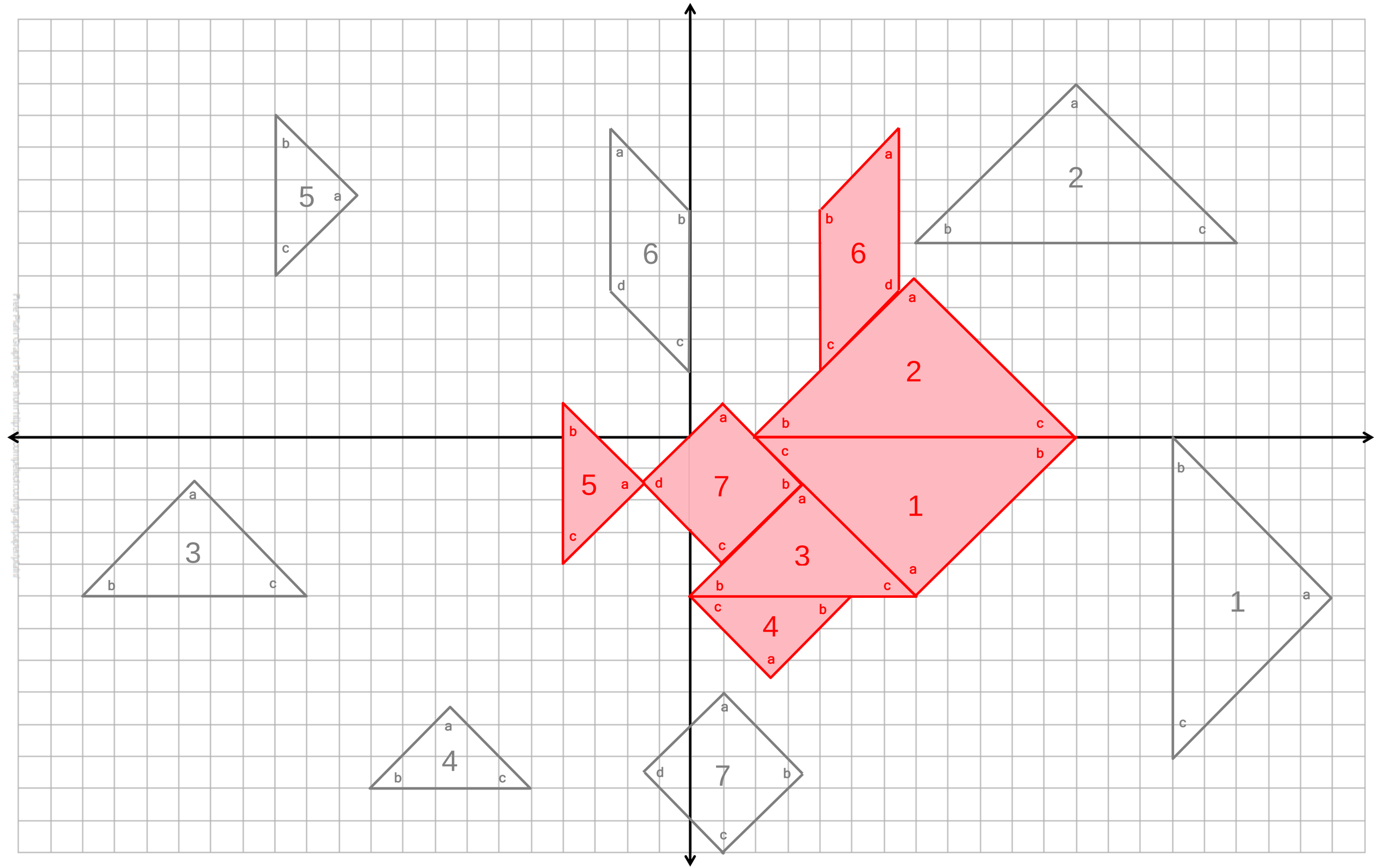








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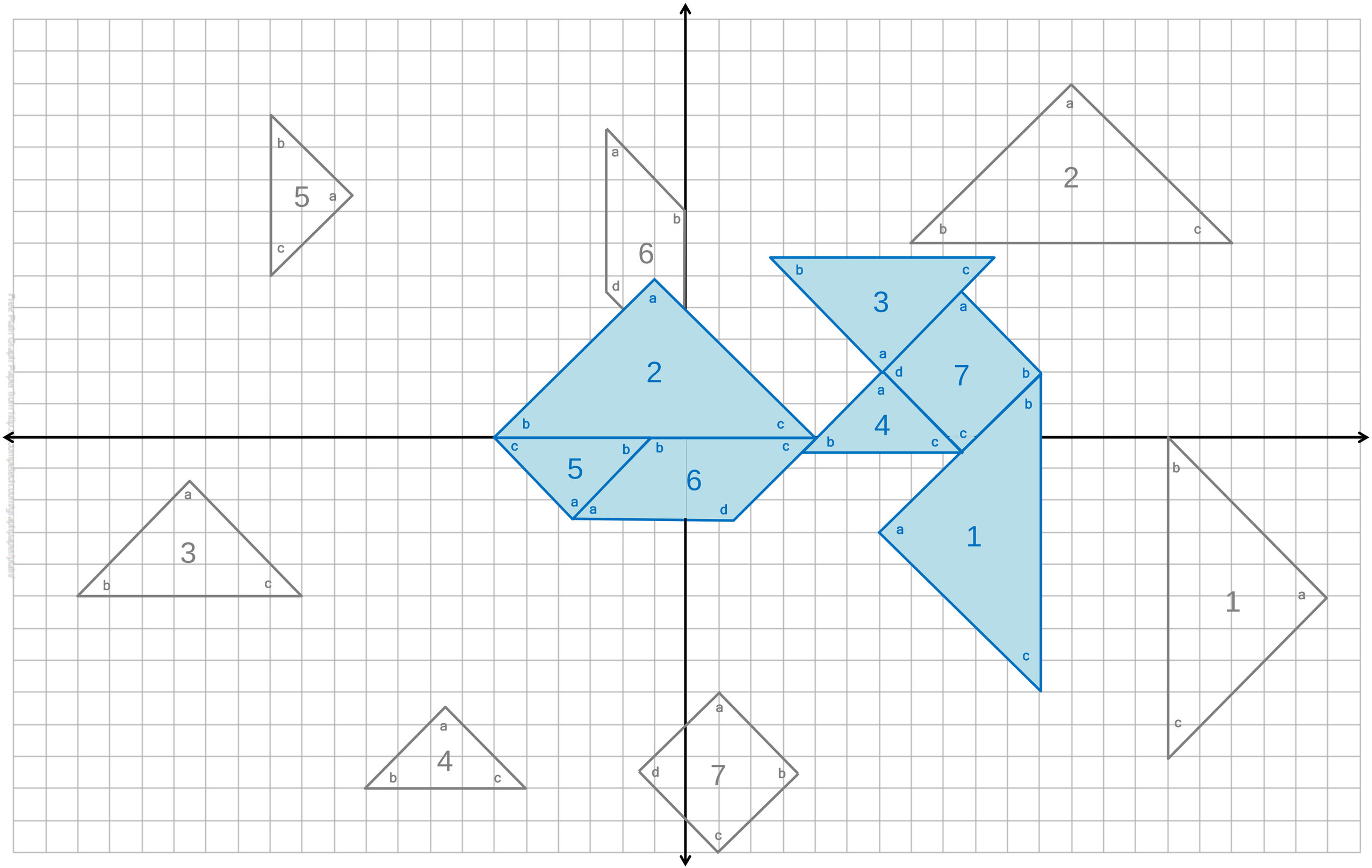
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3
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Transformation Tangrams
Red Example Puzzle Sequences

Identify the sequence of transformations that you used for each tan:

Shape	Transformation(s)
Large Triangle #1	*Rotate 90° CW about the point (15,0). Translate 13 unites to the left.*
Large Triangle #2	*Translate 6 units down and 5 units to the left.*
Medium Triangle #3	*Translate 19 units to the right.*
Small Triangle #4	*Reflect over the line y = -8. Translate 10 units to the right.*
Small Triangle #5	*Translate 9 units down and 9 units to the right.*
Parallelogram #6	*Reflect over the y-axis. Translate 4 units to the right.*
Square #7	*Reflect over the line y = -6.*

Transformation Tangrams
Blue Example Puzzle Sequences

Identify the sequence of transformations that you used for each tan:

Shape	Transformation(s)
Large Triangle #1	*Reflect over the line x = 13. Translate 2 units up.*
Large Triangle #2	*Translate 6 units down and 13 units to the left.*
Medium Triangle #3	*Rotate 180° CW about the origin.*
Small Triangle #4	*Translate 13.5 units to the right and 10.5 units up.*
Small Triangle #5	*Rotate 90° CW about the point (-13,0). Translate 2 units to the left.*
Parallelogram #6	*Rotate 90° CCW about the origin. Translate 6 units to the right.*
Square #7	*Translate 7.5 units to the right and 12.5 units up.*