The Erdős-Lovász Tihany Conjecture for quasi-line graphs

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Abstract

Erdős and Lovász conjectured in 1968 that for every graph G with $\chi(G) > \omega(G)$ and any two integers $s, t \geq 2$ with $s + t = \chi(G) + 1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. Except for a few cases, this conjecture is still unsolved. In this note we prove the conjecture for quasi-line graphs and for graphs with independence number 2.

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1 Introduction

In this paper we consider finite, simple, undirected graphs. Given a graph G, we write $n(G)$ for the number of vertices of G, $\alpha(G)$ for its independence number, $\omega(G)$ for its clique number, $\chi(G)$ for its chromatic number, and $\alpha'(G)$ for the size of a largest matching in G. We write $[r]$ for the set $\{1, \ldots, r\}$. We use the convention that "$A :=$" means that A is defined to be the righthand side of the relation.

A graph is a *quasi-line graph* if the vertex set of the neighborhood of every vertex can be covered by two cliques. By definition, quasi-line graphs are claw-free. Recently, quasi-line graphs attracted more attention (see [2, 3, 4]). In particular, Chudnovsky and Seymour [4] gave a constructive characterization of quasi-line graphs.

Definition 1.1 *A graph G is (s,t)-splittable if $V(G)$ can be partitioned into two sets S and T such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. For $2 \leq s \leq \chi(G) - 1$, we say that G is s-splittable if G is $(s, \chi(G) - s + 1)$-splittable.*

In 1968, Erdős [5] published the following conjecture of Lovász, which has since been known as the 'Erdős-Lovász Tihany Conjecture' (see Problem 5.12 in [6]).

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Conjecture 1 *For every integer $s \geq 2$, every graph G with $\chi(G) > \max\{\omega(G), s\}$ is s-splittable.*

Conjecture 1 is hard, and few related results are known. The only cases of this conjecture that have been settled are $(s,t) \in \{(2,2),(2,3),(2,4),(3,3),(3,4),(3,5)\}$ (see [1, 8, 10, 11]). Recently, Kostochka and Stiebitz [7] proved the conjecture for graphs that are line graphs of (multi)graphs. Here we go one step further: we prove it for quasi-line graphs (in a bit stronger form).

Theorem 1.2 *Let s and t be integers such that $2 \leq s \leq t$. Let G be a quasi-line graph with $\chi(G) = s + t - 1 > \omega(G)$. Then G contains an s-clique S such that $\chi(G - S) \geq t$. In particular, G is s-splittable.*

We also resolve the conjecture for graphs with independence number 2.

Theorem 1.3 *Let $s \geq 2$ be an integer. Let G be a graph with $\alpha(G) = 2$ and $\chi(G) > \max\{\omega(G), s\}$. Then G is s-splittable.*

Note that we cannot hope to prove the strengthening of the Conjecture from Theorem 1.2 for graphs with independence number 2, since such graphs cannot be guaranteed to contain an s-clique for every s.

2 Some lemmas

In this section we present some easy statements. Most of them are known, but for the sake of self-completeness we provide proofs for some of them.

Observation 2.1 *If G is a graph with independence number 2, then*

$$\chi(G) = n(G) - \alpha'(\overline{G}).$$

Let $o(H)$ denote the number of odd components in the graph H.

Theorem 2.2 (Berge-Tutte Formula) *For every graph G,*

$$\alpha'(G) = \min_{P \subset V(G)} \left\{ \frac{n(G) - o(G-P) + |P|}{2} \right\}.$$

Observation 2.1 and the Berge-Tutte Formula immediately yield the following.

Observation 2.3 *If G is a graph with independence number 2, then*

$$\chi(G) = \max_{P \subset V(G)} \left\{ \frac{n(G) + o(\overline{G} - P) - |P|}{2} \right\}.$$

Lemma 2.4 *Let s and t be positive integers. Let G be a graph with*

$$\omega(G) < \chi(G) = s + t - 1$$

and $n(G) \geq \alpha(G)(\chi(G) - 1) + 2$. Then G is s-splittable.

Proof. Let S be any set of $(s-1)\alpha(G)+1$ vertices of G. Then trivially $\chi(G[S]) \geq s$. Furthermore, we have

$$n(G-S) \geq \alpha(G)(\chi(G)-1)+2-(s-1)\alpha(G)-1=(t-1)\alpha(G)+1,$$

which implies $\chi(G-S) \geq t$. □

The remaining statements in this section were proved by Stiebitz [10, 11] long ago.

Lemma 2.5 *Let G be a graph with a clique S of order s such that $\chi(G)=s+t-1$. If G is not s-splittable, then every color class of a $(t-1)$-coloring of $G-S$ contains a vertex adjacent to every vertex of S.*

Proof. Suppose otherwise, and let C be a color class of a $(t-1)$-coloring of $G-S$ containing no vertex adjacent to all vertices of S. Define a coloring of $V(S) \cup C$ by giving each vertex of S a different color and each vertex of C the color of one of its non-neighbors in S. This coloring demonstrates that $V(S) \cup C$ is s-colorable. Also, $G-S-C$ can be colored with $\chi(G-S)-1 \leq t-2$ colors. Therefore $\chi(G) \leq s+t-2$, a contradiction. □

Lemma 2.5 immediately implies the following.

Corollary 2.6 *If G has a maximal clique of order s, then G is s-splittable. In particular, every graph G is s-splittable with $s=\omega(G)$.* □

3 Quasi-Line Graphs: Proof of Theorem 1.2

For the proof of Theorem 1.2, we consider a counterexample G with the fewest edges. Our strategy is to consider an s-clique in G and, in a series of lemmas, find an $(s+t-1)$-clique containing it, contradicting the condition that $\omega(G) < \chi(G)$.

Edge-minimality of G implies that it is $(s+t-1)$-critical. Therefore, G is connected and $\delta(G) \geq s+t-2$. Since Conjecture 1 holds for complete graphs and odd cycles, by Brooks's theorem, $\Delta(G) \geq s+t-1$. Consider a vertex x of maximum degree in G. By the definition of quasi-line graphs, $N(x)$ can be written as $A \cup B$ where $G[A]$ and $G[B]$ are complete graphs. Since $|A|+|B| \geq s+t-1$ and $s \leq t$, we have $\max\{|A|,|B|\} \geq s$. In particular, we know that $\omega(G) \geq s$. To arrive at a contradiction, we will show that $\omega(G)=s+t-1$.

Let $\mathcal{P}$ denote the set of all pairs (S,f) such that S is a s-clique in G and f is a proper $(t-1)$-coloring of $G-S$. Since G is not s-splittable, $\chi(G-S)=t-1$ for each s-clique S of G, and hence $\mathcal{P}$ is nonempty. For a color $k \in [t-1]$, let

$$C_k(f) := \{y \in V(G) \setminus S \mid f(y)=k\}$$

denote the corresponding color class in f.

Consider an arbitrary pair $(S,f) \in \mathcal{P}$. Define a digraph $D := D(S,f)$ as follows. The vertex set of D is $V(G)$. For vertices $x,y \in V(G)$, we let $xy \in E(D)$ if and only if $xy \in E(G)$, $y \in V(G) \setminus S$ and y is the unique neighbor of x in the set $C_{f(y)}(f)$ in the graph G. For a vertex $x \in S$, let $R_x(S,f)$ denote the set of all vertices y such that D contains a directed path P from x to y. Furthermore, let

$$R(S,f) := \bigcup_{x \in S} R_x(S,f).$$

Definition 3.1 *Let S be a clique of order s in G. Let $x \in S$, $y \in V(G) \setminus S$, and y is adjacent to every vertex of S. Let f be a $(t-1)$-coloring of $G - S$.*

Let S' be the clique $(S \setminus \{x\}) \cup \{y\}$ and f' be the coloring of $G - S'$ defined by $f'(v) = f(v)$ if $v \in V(G - S - y)$ and $f'(x) = f(y)$.

The pair (S', f') will be denoted by $(S, f)/xy$.

Lemma 3.2 *Let $(S, f) \in \mathcal{P}$ and let xy be an edge of $D = D(S, f)$ such that $x \in S$. Then y is adjacent in G to every vertex of S. Furthermore, the pair $(S', f') = (S, f)/xy$ belongs to $\mathcal{P}$.*

Proof. Let S, f, x, and y be as in the statement. By Lemma 2.5, there is a vertex u with $f(u) = f(y)$ which is adjacent in G to every vertex of S. Since $x \in S$ and y is the only neighbor of x in the set $C_{f(y)}(f)$, we have $u = y$. This gives the first half of the lemma. Since $S' = (S \setminus \{x\}) \cup \{y\}$ is an s-clique in G and the map f' is a $(t-1)$-coloring of $G - S'$, we have $(S', f') \in \mathcal{P}$. □

Lemma 3.3 *Let $(S, f) \in \mathcal{P}$ and let xy be an edge of $D = D(S, f)$ such that $x \in S$. Then, for each color $k \in [t-1]$ with $k \neq f(y)$, there is a vertex z such that $f(z) = k$ and z is adjacent in G to every vertex of $S \cup \{y\}$.*

Proof. By Lemma 3.2, y is adjacent to every vertex of S. Let $\ell = f(y)$ and fix a color $k \in [t-1] \setminus \{\ell\}$. By Lemma 2.5, there is a vertex $w \in C_k(f)$ that is adjacent in G to every vertex of S. If $yw \in E(G)$, then we are done, so suppose that $yw \notin E(G)$. By Lemma 3.2, the pair $(S', f') := (S, f)/xy$ is in the set $\mathcal{P}$. Now Lemma 2.5 tells us there is a vertex $v \in C_k(f')$ adjacent in G to every vertex of $S' = (S \setminus \{x\}) \cup \{y\}$. Since $yw \notin E(G)$, we know that $v \neq w$. Since $f(v) = k = f(w)$, we see $vw \notin E(G)$.

Since $s \geq 2$, there is a vertex $x' \in S \setminus \{x\}$. First, we claim that $x'y \in E(D)$. Otherwise, there is a neighbor y' of x' with $y' \neq y$ and $f(y') = f(y) = \ell$. Since y is the only neighbor of x with color ℓ and $f(y') = \ell$, we have $y'x \notin E(G)$. We also know that $y'y \notin E(G)$ since both vertices are in $C_\ell(f)$. Hence the sequence $X = (x, y', y, w, v, x)$ is the complement of a 5-cycle in the neighborhood of x' in G, and so the neighborhood of x' cannot be covered by two cliques, a contradiction. Therefore $x'y \in E(D)$.

By Lemma 3.2, the pair $(S^*, f^*) := (S, f)/x'y$ is in the set $\mathcal{P}$. It follows from Lemma 2.5 that there is a vertex $u \in C_k(f^*)$ adjacent to every vertex of $S^* = (S \setminus \{x'\}) \cup \{y\}$. Since yw and vx are not edges in G, $u \notin \{v, w\}$. Observe that since $C_k(f^*) = C_k(f)$, we know uw and uv are not edges of G. If $s \geq 3$, then there is a vertex $x'' \in S \setminus \{x, x'\}$, and $\{u, v, w\}$ is an independent set of order three in the neighborhood of x''. This contradicts the fact that G is claw-free. Thus we have already proved the lemma for $s \geq 3$.

It remains to prove the lemma in the case $s = 2$. Let $H_1 := G[S \cup C_k(f) \cup C_\ell(f)]$. Obviously, $\chi(H_1) = s + 2 = 4$. The subgraph $H_2 = G[\{x, x', y, u, v, w\}]$ (see Figure 1) has a 3-coloring h defined by $h(u) = h(x') = 1, h(x) = h(v) = 2$, and $h(w) = h(y) = 3$. We claim that this 3-coloring of H_2 can be extended to a 3-coloring of H_1, which contradicts that $\chi(H_1) = s + 2 = 4$.

Let $Y := C_\ell(f) - \{y\}$. Since xy and $x'y$ are edges of D, there is no edge in G from $\{x, x'\}$ to Y. Combining this with the fact that $y \in C_\ell(f)$, we conclude that G has no edges from $\{x, x', y\}$ to Y.

Since $U = \{u, v, w\}$ is an independent set in the claw-free graph G, U cannot be contained in the neighborhood of any vertex in G. Therefore, every vertex $y' \in Y$ has a non-neighbor $g(y') \in U$.

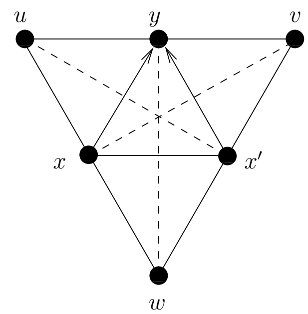

Figure 1: The subgraph $G[\{x, x', y, u, v, w\}]$.

Thus we can extend the coloring h of H_2 to a 3-coloring h' of $G[\{x, x'\} \cup U \cup C_\ell(f)]$ by coloring each $y' \in Y$ with $h(g(y'))$.

Now we extend the coloring h' to include the set $Z := C_k(f) \setminus U$. Since G is claw-free, no vertex has three neighbors in any color class of f. Since $\{xw, xu, x'w, x'v, yv, yu\} \subseteq E(G)$, none of x, x', and y has a neighbor in Z. Also, no vertex of Z has three neighbors in $C_\ell(f)$. We conclude that G has no edges between Z and $\{x, x', y\} \cup U$ and that each vertex of Z has at most two neighbors in Y. Hence for each $z \in Z$, there is a color of h' not used in the neighborhood of z, so we can extend the 3-coloring h' to include Z. This coloring shows that $\chi(H_1) \leq 3$, a contradiction. This completes the proof. □

The following simple observation will be used several times.

Observation 3.4 *Let (S, f) be a pair in $\mathcal{P}$ such that the vertex $x \in S$. Let $D = D(S, f)$. Suppose that $xy \in E(D)$. Let $(S', f') = (S, f)/xy$ and let $D' = D(S', f')$. If $zu \in E(D) \setminus E(D')$ and $\{z, u\} \cap \{x, y\} = \emptyset$, then $zy \notin E(G)$ and $zx \in E(G)$.*

Lemma 3.5 *Let $(S, f) \in \mathcal{P}$ such that $x_0 \in S$. Let $D = D(S, f)$ and $P := (x_0, x_1, \dots, x_p)$ be a directed path in D. Then $R := S \cup V(P)$ induces a clique in G.*

Proof. Suppose that the lemma is false, and let p be the smallest positive integer for which the lemma is false. By Lemma 3.2, we know that $p \geq 2$.

Let (S, f), D, and x_0 be as in the statement of the lemma and let $P := (x_0, x_1, \dots, x_p)$ be such that $R := S \cup V(P)$ does not induce a clique in G. By the minimality of p, $S \cup \{x_0, \dots, x_{p-1}\}$ induces a clique in G. In particular, if $p \geq 3$, then $x_1x_{p-1} \in E(G)$.

Let $(S', f') := (S, f)/x_0x_1$ and $D' := D(S', f')$. Lemma 3.2 tells us that $(S', f') \in \mathcal{P}$. Since $G[\{x_0, \dots, x_{p-1}\}]$ is a clique, Observation 3.4 implies that D' contains the directed path $(x_1, \dots, x_p)$. By the minimality of p, $G[(S \setminus \{x_0\}) \cup \{x_1, \dots, x_p\}]$ is a clique. Therefore, if $x_0x_p \in E(G)$, then we are done, so assume that $x_0x_p \notin E(G)$.

Since $G[\{x_1, \dots, x_p\}]$ is a clique, the colors $f(x_1), \dots, f(x_p)$ are pairwise distinct. By Lemma 3.3, there is a vertex y_p with $f(y_p) = f(x_p)$ such that y_p is adjacent in G to every vertex of $S \cup \{x_1\}$. Since $y_px_0 \in E(G)$, we know that $y_p \neq x_p$. Furthermore, $x_py_p \notin E(G)$ because both vertices are in $C_{f(x_p)}(f)$, and $x_{p-1}y_p \notin E(G)$ since $x_{p-1}x_p \in E(D)$. (Note that for $p = 2$ this is already a contradiction.)

Since $s \geq 2$, there is a vertex $x \in S \setminus \{x_0\}$. Observe that x is adjacent in G to every vertex of $S \cup V(P) \cup \{y_p\}$. We claim that for every $k \in [p-1]$, there is a vertex $y_k \in N_G(x) \setminus \{x_k\}$ such that

$f(y_k) = f(x_k)$. Suppose that this is false for some $k \in [p-1]$. Then x_k is the unique neighbor of x in G with color $f(x_k)$, and so $xx_k \in E(D)$. If $k \geq 2$, then $P'_k := (x, x_k, \ldots, x_p)$ is a directed path in D shorter than P. By the minimality of p, $G[S \cup V(P'_k)]$ is a clique. In particular, $x_0x_p \in E(G)$, a contradiction. Let $k = 1$. Consider $(S^*, f^*) := (S, f)/xx_1$ and $D^* = D(S^*, f^*)$. As in the previous paragraph, we conclude that $(S^*, f^*) \in \mathcal{P}$ and D^* contains the directed path $(x_1, \ldots, x_p)$. So, by the minimality of p, $G[(S \setminus \{x\}) \cup \{x_1, \ldots, x_p\}]$ is a clique, and hence $x_0x_p \in E(G)$, a contradiction. This proves the existence of y_k for every $k \in [p-1]$.

Note that for each $k \in [p]$, $x_{k-1}y_k$ is not an edge of G because x_k is the unique neighbor in G of x_{k-1} with color $f(x_k) = f(y_k)$. This implies that $(x_0, y_1, x_1, y_2, x_2, \ldots, y_{p-1}, x_{p-1}, y_p, x_p)$ is the complement of an odd cycle in the neighborhood of x, which is impossible since G is a quasi-line graph. This contradiction completes the proof. □

Lemma 3.6 *Let $(S, f) \in \mathcal{P}$. Then $R(S, f)$ induces a clique in G.*

Proof. Let x and y be any two distinct vertices in the set $R(S, f)$. We need to prove that $xy \in E(G)$.

By the definition of $R(S, f)$, there are directed paths $P = (x_0, \ldots, x_p)$ and $Q = (y_0, \ldots, y_q)$ in $D(S, f)$ such that x_0 and y_0 are vertices of S and that $x_p = x$ and $y_q = y$. We may assume without loss of generality that $q \leq p$.

We prove the lemma by induction on $p + q$, and subject to that, by induction on q. If $q = 0$, we are done by Lemma 3.5. If $p = q = 1$, then we are done by applying Lemma 3.3 and making use of fact that y_1 is the only vertex of its color adjacent to y_0. We may assume, then, that $p \geq 2$.

By the minimality of $p + q$,

$$\text{each } y' \in \{y_0, \ldots, y_q\} \text{ is adjacent to each } x' \text{ in } \{x_0, \ldots, x_{p-1}\} \setminus \{y'\}. \tag{1}$$

Let $(S', f') := (S, f)/y_0y_1$ and $D' = D(S', f')$. We consider three cases.

CASE 1: $\{y_0, y_1\} \cap V(P) = \emptyset$. By (1) and Observation 3.4, P and $Q' := Q - y_0$ are directed paths in D' the sum of lengths of which is $p + q - 1$. Since $y_1 \in S'$, minimality of $|V(P)| + |V(Q)|$ yields that $xy \in E(G)$, as required.

CASE 2: $\{y_0, y_1\} \cap V(P) = \{y_0\}$. Then $y_0 = x_0$. Again by (1) and Observation 3.4, $P' = (y_1, x_0, x_1, \ldots, x_p)$ and $Q' := Q - y_0$ are directed paths in D'. Now $|V(P')| + |V(Q')| = p + q$, but since $|V(Q')| < |V(Q)|$, the secondary induction assumption implies that $xy \in E(G)$.

CASE 3: $y_1 \in V(P)$. Suppose $y_1 = x_k$. If $k \geq 2$, then we are done by the induction assumption applied to $P^* = (y_0, x_k, x_{k+1}, \ldots, x_p)$ and Q in D. Let $k = 1$. Then we are done by the induction assumption applied to $P'' = (x_1, x_2, \ldots, x_p)$ and $Q' := Q - y_0$ in D'. □

Lemma 3.7 *Let $(S, f) \in \mathcal{P}$ and let $D := D(S, f)$. Let $P := (x_0, x_1, \ldots, x_p)$ be a directed path in D such that $x_0 \in S$. Then, for each color*

$$k \in [t-1] \setminus \{f(x_1), \ldots, f(x_p)\},$$

there is a vertex $z \in C_k(f)$ adjacent in G to every vertex of $S \cup V(P)$.

Proof. We prove the lemma by induction on p. For $p = 0$, the statement is Lemma 2.5, and for $p = 1$, it is Lemma 3.3. We assume, then, that $p \geq 2$.

By Lemma 3.5, $R := S \cup V(P)$ induces a clique in G. Therefore, the colors $f(x_1), \ldots, f(x_p)$ are pairwise distinct. Let k be an arbitrary color in the set $[t-1] \setminus \{f(x_1), \ldots, f(x_p)\}$. By the induction hypothesis, there is a vertex $y_p^k \in C_k(f)$ adjacent in G to every vertex of $S \cup \{x_1, \ldots, x_{p-1}\}$. If $y_p^k x_p \in E(G)$ then the statement is proved, so assume $y_p^k x_p \notin E(G)$.

Consider the pair $(S', f') := (S, f)/x_0x_1$. Then $S' = (S \setminus \{x_0\}) \cup \{x_1\}$ induces an s-clique in G, the pair (S', f') is in $\mathcal{P}$, and by Observation 3.4, we know that $P' := (x_1, x_2, \ldots, x_p)$ is a directed path in $D' := D(S', f')$. By the induction hypothesis, there is a vertex $z_p^k \in C_f(k)$ adjacent to every vertex of $S' \cup \{x_1, \ldots, x_{p-1}\}$. If $z_p^k x_0 \in E(G)$, then we are done, so again assume that $z_p^k x_0 \notin E(G)$. Then we know that $z_p^k \neq y_p^k$ since $y_p^k x_0 \in E(G)$ and that $z_p^k y_p^k \notin E(G)$ since $f(z_p^k) = f(y_p^k)$.

Since $s \geq 2$, there is a vertex $x \in S \setminus \{x_0\}$. So far, we have that x is adjacent to every vertex of $(S \setminus \{x\}) \cup V(P) \cup \{y_p^k, z_p^k\}$.

As in the proof of Lemma 3.5, we claim that, for every $\ell \in [p]$, there is a vertex $v_\ell \in N_G(x) \setminus \{x_\ell\}$ such that $f(v_\ell) = f(x_\ell)$. Suppose that this is false for some $\ell \in [p]$. Then x_ℓ is the unique neighbor in G of x in the set $C_{f(x_\ell)}(f)$ and, therefore, $xx_\ell \in E(D)$. Then $P^* := (x, x_\ell, \ldots, x_p)$ is a directed path in D with $x \in S$. If $\ell \geq 2$, then, by the minimality of p, there is a vertex $u_k \in C_k(f)$ adjacent in G to all vertices in $S \cup V(P^*)$. Since $y_p^k x_p \notin E(G)$, we know that $u_k \neq y_p^k$, and since $z_p^k x_0 \notin E(G)$, we have $u_k \neq z_p^k$. Then x is adjacent to three distinct vertices, namely u_k, y_p^k, and z_p^k, of color k, which contradicts that G is a quasi-line graph. If $\ell = 1$, consider the pair $(S^*, f^*) = (S, f)/xx_1$ and the directed path $P' := (x_1, x_2, \ldots, x_p)$ in $D(S^*, f^*)$. By the minimality of p, there is a vertex $u_k \in C_k(f)$ adjacent to all vertices in $S^* \cup V(P')$. Since $x_0, x_p \in S^* \cup V(P')$, as before, we have that $u_k \notin \{z_p^k, y_p^k\}$. Recall that $p \geq 2$, so x_1 is adjacent to 3 distinct vertices of color k, a contradiction.

Summarizing, for every $\ell \in [p]$ there is a $v_\ell \in C_{f(v_\ell)}(f) - \{x_\ell\}$ such that v_ℓ is adjacent to x. Since $x_{\ell-1}x_\ell \in D$, we have $x_{\ell-1}v_\ell \notin E(G)$. Therefore, the sequence $(x_0, v_1, x_1, v_2, x_2, \ldots, v_{p-1}, x_{p-1}, v_p, x_p, y_p, z_p, x_0)$ is the complement of an odd cycle in the $N_G(x)$. Since G is a quasi-line graph, this is a contradiction, and the lemma is proved. □

Lemma 3.8 *Let $(S, f) \in \mathcal{P}$ and $C := \{f(v) \mid v \in R(S, f) \setminus S\}$. Then, for each color $k \in [t-1] \setminus C$, there is a vertex z such that $f(z) = k$ and z is adjacent in G to every vertex of $R(S, f)$.*

Proof. Suppose that there is a color $k \in [t-1] \setminus C$ such that no vertex in $C_k(f)$ is adjacent in G to every vertex of $R(S, f)$. Then Lemma 2.5 implies that $R(S, f) \setminus S$ is nonempty. Let $D = D(S, f)$ and let x_1 be a vertex of $R(S, f) \setminus S$.

By Lemma 3.7, there is a vertex $z_1 \in C_k(f)$ adjacent in G to every vertex of $S \cup \{x_1\}$. By our assumption, there is an $x_2 \in R(S, f)$ such that $x_2z_1 \notin E(G)$. Again by Lemma 3.7, there is a vertex $z_2 \in C_k(f)$ adjacent in G to every vertex of $S \cup \{x_2\}$. Since $x_2z_1 \notin E(G)$, we know that $z_2 \neq z_1$. By construction, each vertex in S is adjacent to both z_1 and z_2. Let y be the closest vertex to S in the graph D such that is $y \notin N_G(z_1) \cap N_G(z_2)$. By symmetry, we may assume that $yz_2 \notin E(G)$ (it is possible that $y = z_1$). Let P be a shortest path in D from S to y, and write $P == (y_0, \ldots, y_p)$ where $y = y_p$.

Write $(S_0, f_0) := (S, f)$, and for $i = 1, \ldots, p$, let $(S_i, f_i) := (S_{i-1}, f_{i-1})/y_{i-1}y_i$. Since, according to Lemma 3.6, $R(S, f)$ induces a clique in G, by Observation 3.4, the pair (S_i, f_i) is in $\mathcal{P}$ for each $i \in [p]$. By construction, $S_p = (S \setminus \{y_0\}) \cup \{y\}$. Again by Lemma 3.7, there is a vertex $z_3 \in C_k(f)$ adjacent in G to every vertex of $S_p \cup \{x_2\}$ (recall that by the choice of y, $x_2 \notin V(P)$). Since z_3 is adjacent to both x_2 and y_p, we see that $z_3 \notin \{z_1, z_2\}$. Then each vertex $v \in S \setminus \{y_0\}$ is adjacent to three vertices, namely z_1, z_2, and z_3, in $C_k(f)$, which contradicts to the fact that G is claw-free. □

After all this preparation, we turn to the proof of the theorem:

Consider a pair $(S, f) \in \mathcal{P}$ and let $R := R(S, f)$. Define the sets $C_1 := \{f(v) \mid v \in R \setminus S\}$ and $C_2 := [t-1] \setminus C_1$. Lemma 3.6 implies that $G[R]$ is a complete graph, and so $|C_1| = |R \setminus S|$. This implies that $|C_2| = s + t - 1 - |R|$. Since $\omega(G) < s + t - 1$, C_2 is non-empty.

It follows from Lemma 3.8 that for each color $k \in C_2$, there is a vertex $z_k \in C_k(f)$ such that z_k is adjacent in G to every vertex of R. It follows that $\omega(G) \geq |R| + 1$.

Let $q := |C_2|$, and assume, without loss of generality, that $C_2 = [q]$. Let $V_2 := \{v \in V(G) : f(v) \in C_2\}$. For every $k \in [t-1]$ and $v \in V(G)$, let $C_k(f, v) := N(v) \cap C_k(f)$. By the definition of R, for every $v \in R$ and every $k \in C_2$, we have that

$$|C_k(f, v)| = 2, \tag{2}$$

for otherwise, if $C_k(f, v)$ consisted of a single vertex, then that vertex would be in R, forcing $k \notin C_2$.

Fix a vertex $x \in S$ and let $W := N(x) \cap (R \cup V_2)$. Since G is a quasi-line graph, the complement, $\overline{G[W]}$, of $G[W]$ is bipartite. For each $k \in C_2$, let a_k and b_k be the vertices of color k in W. Let $M := \{a_k b_k : k \in C_2\}$. If M is a maximum matching in $\overline{G[W]}$, then by the König-Egerváry Theorem, $\overline{G[W]}$ has an independent set of size $|R| - 1 + |C_2|$, and this set together with x is a clique in G of order at least $s + t - 1$, a contradiction. Thus, $\overline{G[W]}$ has a matching larger than M.

By Berge's Theorem, there is an M-augmenting path in $\overline{G[W]}$. Choose a shortest such path P, and write $P = (y, a_1, b_1, a_2, \ldots, b_p, z)$ where $y, z \in R$. We know that $p \geq 1$, as $yz \in E(G)$. Since $\overline{G[W]}$ is bipartite, we have $yb_i \notin E(\overline{G[W]})$ for each $i \in [p]$. The fact that P is a shortest path implies that $ya_i \notin E(\overline{G[W]})$ for all $i \geq 2$. Similarly, $zb_i \notin E(\overline{G[W]})$ for $i \in [p-1]$ and $za_i \notin E(\overline{G[W]})$ for $i \in [p]$.

By Lemma 3.8, there is a vertex w of color 1 adjacent to every vertex of R. Since $wx \in E(G)$, we know that $w \in \{a_1, b_1\}$. We conclude that $w \neq a_1$, that is, that $w = b_1$, as $wy \in E(G)$. In particular,

$$zb_1 \in E(G). \tag{3}$$

(Note that we already knew this in the case $p \geq 2$.) Since $ya_1 \in E(\overline{G[W]})$, by (2), y has a neighbor $d_1 \notin W$ in G of color 1. By (2), we can see that $zd_1 \notin E(G)$, as $za_1, zb_1 \in E(G)$. So, $(d_1, b_1, a_2, \ldots, b_p, z, d_1)$ is an odd cycle in the complement of $N(y)$. This contradiction establishes Theorem 1.2. □

4 Independence number 2: Proof of Theorem 1.3

Lemma 4.1 *Let G be a graph with $\alpha(G) = 2$ and $\omega(G) < \chi(G) = s + t - 1$. If $\omega(G) \geq s$, then G is (s, t)-splittable.*

Proof. Write n for the order of G. Let S_0 be a set of s vertices inducing a clique in G, and let $T_0 = V(G) - S_0$. If $\chi(G[T_0]) \geq t$, then we are done. Otherwise, since $\alpha(G) = 2$, we have

$$t - 1 \geq \chi(G[T_0]) \geq (n - |S_0|)/2 = (n - s)/2.$$

Adding s to both sides, we get

$$\chi(G) = s + (t - 1) \geq (n + s)/2. \tag{4}$$

By Observation 2.3, there is a $P \subset V(G)$ such that

$$\chi(G) = \frac{n + o(\overline{G} - P) - |P|}{2}. \tag{5}$$

Choose a largest such set P, so that every component of $\overline{G} - P$ is odd. Combining (5) and (4), we get

$$o(\overline{G} - P) \geq s - 1. \tag{6}$$

If every component of $\overline{G} - P$ consists of a single vertex, then $\omega(G) \geq n - |P|$. In addition, we have $o(\overline{G} - P) = n - |P|$, and so (5) shows $\chi(G) = n - |P| \leq \omega(P)$, a contradiction. Therefore, we assume that some component, call it H, of $\overline{G} - P$ has at least 3 vertices.

Since $\overline{G}$ is triangle-free, H contains a pair $\{x, y\}$ of non-adjacent vertices. By (6), we can choose a set S' of $s - 2$ vertices, each from a different component of $\overline{G} - P - V(H)$. Let $S = \{x, y\} \cup S'$. By construction, S induces an s-clique in G. Since $|V(H) - \{x, y\}|$ is odd, $o(\overline{G} - S - P) \geq o(\overline{G} - P) - (s - 2)$. Hence, by Observation 2.3,

$$\begin{aligned} \chi(G - S) &\geq \frac{(n - s) + o(\overline{G} - S - P) - |P|}{2} \\ &\geq \frac{n - s + o(\overline{G} - P) - (s - 2) - |P|}{2} \\ &= \chi(G) - s + 1 = t. \end{aligned}$$

This certifies that G is (s, t)-splittable. □

Proof of Theorem 1.3: Let G be a counterexample to the theorem. In light of Lemma 4.1,

$$\omega(G) \leq s - 1. \tag{7}$$

As always, we assume that $s \leq t$, and so $\chi(G) \geq 2s - 1$. If $n \leq 3s - 2$, then in each $(s + t - 1)$-coloring of G, at least $2(s + t - 1) - n \geq 4s - 2 - (3s - 2) = s$ color classes consist of only one vertex. The vertices of these color classes contain an s-clique, which contradicts (7). From now on, then, we assume $n \geq 3s - 1$.

As in the proof of Lemma 4.1, choose $P \subseteq V(G)$ satisfying (5) of maximum size, so that each component of $\overline{G} - P$ has an odd order. Note that $o(\overline{G} - P) - |P| \geq 0$ by Observation 2.3.

If $o(\overline{G} - P) - |P| = 0$, then $\chi(G) = n/2$, so G is s-splittable by Lemma 2.4. Hence, from now on, we assume that $|P| \leq o(\overline{G} - P) - 1$. Since $o(\overline{G} - P) \leq \omega(G)$, by (7) we get $|P| \leq s - 2$.

CASE 1: $0 < |P| \leq s - 2$. Let the set X contain exactly one vertex from each component of $\overline{G} - P$. From (7) we see $|X| \leq s - 1$. For each component H of $\overline{G} - P$, we know that $|V(H) - X|$ is even. This along with the fact that $n \geq 3s - 1$ tells us that we can find a $2(s - \lceil |P|/2 \rceil)$-element subset $S' \subset V(\overline{G} - P - X)$ that has an even number of vertices in common with each component of $\overline{G} - P$ (we can construct S' greedily by adding pairs from components of $G - P - X$).

Let $S = S' \cup P$. Then

$$|S| = 2s - 2\lceil |P|/2 \rceil + |P| \geq 2s - 1.$$

Hence $\chi(G[S]) \geq s$. On the other hand, since each component H of $\overline{G} - P$ satisfies that $|V(H)| - S'$ is odd, by Observation 2.3 (with $P = \emptyset$),

$$\chi(G - S) \geq \frac{n - |S| + o(\overline{G} - S)}{2}$$

$$
\begin{aligned}
&= \frac{n - |S| + o(\overline{G} - P)}{2} \\
&= \frac{n + o(\overline{G} - P) - |P|}{2} - s + \lceil |P|/2 \rceil \\
&\geq \chi(G) - s + 1 = t.
\end{aligned}
$$

This certifies that G is s-splittable.

CASE 2: $P = \emptyset$. In this case, each component of $\overline{G}$ has an odd order, so $\chi(G) = (n + o(\overline{G}))/2$. We know that $\overline{G}$ has a component H of order at least 3, as $o(\overline{G}) \leq \omega(G) \leq s-1$ and $n \geq 3s-1$. Since $\overline{G}$ is triangle-free, H contains two non-adjacent (in $\overline{G}$) vertices x and y. Let $F = N_{\overline{G}}(x) \cup N_{\overline{G}}(y)$. Since $\overline{G}$ is triangle-free and $\chi(G) \leq s-1$, it follows that $|F| \leq 2(s-1)$.

SUBCASE 2.1: $|V(H)| \geq 2s+1$. Let S be any $(2s-1)$-element subset of $V(H) \setminus \{x, y\}$ that contains F. Then $\chi(G[S]) \geq \lceil |S|/2 \rceil = s$, and (since x and y form components of $\overline{G} - S$) by Observation 2.3,

$$
\chi(G - S) \geq \frac{(n - 2s + 1) + o(\overline{G} - S)}{2} \geq \frac{n - 2s + 1 + o(\overline{G}) + 1}{2} = \chi(G) - s + 1 = t.
$$

SUBCASE 2.2: $|V(H)| \leq 2s - 1$. Let X contain exactly one vertex from each component of $\overline{G} - V(H)$. As in Case 1, since $|H' \setminus X|$ is even for each component H' of $\overline{G} - V(H)$, we can find a $(2s + 1 - |V(H)|)$-element subset $S' \subseteq V(\overline{G} - X - V(H))$ that has an even number of vertices in common with each component of $\overline{G} - V(H)$. Let $S = S' \cup (V(H) \setminus \{x, y\})$. By construction, $|S| = 2s - 1$, and hence $\chi(G[S]) \geq s$. On the other hand, since x and y form odd components of $\overline{G} - S$, by Observation 2.3,

$$
\begin{aligned}
\chi(G - S) &\geq \frac{n - |S| + o(\overline{G} - S)}{2} \\
&\geq \frac{n - 2s + 1 + o(\overline{G}) + 1}{2} \\
&= \frac{n + o(\overline{G})}{2} - s + 1 = \chi(G) - s + 1 = t.
\end{aligned}
$$

Therefore G is s-splittable. □

References

[1] W. G. Brown and H. A. Jung, On odd circuits in chromatic graphs, *Acta Math. Acad. Sci. Hungar.* **20** (1999), 129–134.

[2] M. Chudnovsky and A. Ovetsky, Coloring quasi-line graphs, *J. Graph Theory* **54** (2007), 41–50.

[3] M. Chudnovsky and A. Ovetsky Fradkin, Hadwiger's Conjecture for quasi-line graphs, to appear in *J. Graph Theory.*

[4] M. Chudnovsky and P. D. Seymour, Claw-free graphs VII — Quasi-line graphs, in preparation,

[5] P. Erdős, Problems, Theory of Graphs, Proc. Colloq. Tihany, Academic Press, New York, 1968, 361–362.

[6] T.R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley Interscience, New York, 1995.

[7] A. V. Kostochka and M. Stiebitz, Partitions and edge colorings of multigraphs, *Electron. J. Combin.* **15** (2008), N25.

[8] N. N. Mozhan, On doubly critical graphs with chromatic number five, Technical Report 14, Omsk Institute of Technology, 1986 (in Russian).

[9] C. E. Shannon, A theorem on coloring the lines of a network, *J. Math. Phys.* **28** (1949), 148–151.

[10] M. Stiebitz, K_5 is the only double-critical 5-chromatic graph, *Discrete Math.* **64** (1987), 91–93.

[11] M. Stiebitz, On k-critical n-chromatic graphs. In: Colloquia Mathematica Soc. János Bolyai **52**, Combinatorics, Eger (Hungary), 1987, 509–514.