Geometric Series

Defn: An infinite series is an expression of the form

\[ a_1 + a_2 + a_3 + \ldots + a_k + \ldots \quad \text{or} \quad a_k + \ldots \quad \text{or simply} \quad a_k \]

The problem is to add up an infinite number of terms to find the value of the sum. Sometimes, it's fairly easy:

\[ .333333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots = \frac{1}{3} \]

This type of series may look familiar. It's called a geometric series. Let's first look at the \(n^{th}\) term. Complete the following:

<table>
<thead>
<tr>
<th>Recursive form</th>
<th>Closed (explicit) form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1 = a_1)</td>
<td>(a_1 = a_1)</td>
</tr>
<tr>
<td>(a_2 = a_1r)</td>
<td>(a_2 = a_1r)</td>
</tr>
<tr>
<td>(a_3 = a_2r)</td>
<td>(a_3 = a_1r^2)</td>
</tr>
<tr>
<td>(a_4 = )</td>
<td>(a_4 = )</td>
</tr>
<tr>
<td>(a_n = )</td>
<td>(a_n = )</td>
</tr>
</tbody>
</table>

Sum of a Finite Geometric Series

Let \(S_n = a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1}\)

Multiply each term above by \(r\), and vertically line up terms similar to the terms above, according to the power of \(r\).

\[ r \cdot S_n = \]

Subtract the second line from the first, vertically, according to similar terms.

\[ S_n - r \cdot S_n = \]

Factor the left side and solve for \(S_n\).

\[
\begin{align*}
(1) \quad & \text{Find } \sum_{j=1}^{8} 3^j \\
(2) \quad & \text{Find } \sum_{j=1}^{6} 2 \cdot \left(\frac{1}{3}\right)^j
\end{align*}
\]
Infinite Geometric Series

Given \( S_n = \frac{a_1(1 - r^n)}{1 - r} \), what happens to the formula for \( S_n \) when \( r = 1 \)?

What does the actual series become when \( r = 1 \)? Write out a few terms.

As \( n \to +\infty \) with \( |r| > 1 \), what happens to \( |r^n| \)? Does the series converge or diverge?

As \( n \to +\infty \) with \( |r| < 1 \), what happens to \( r^n \)?

In terms of \( a \) and \( r \), what is \( \lim_{n \to \infty} S_n \)?

Thus, for an infinite geometric series with \( |r| < 1 \) and first term \( a_1 = a \), we find

\[
S = \lim_{n \to \infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a}{1 - r}.
\]

Find:

(3) \[ 6 + 4 + \frac{8}{3} + \ldots \]

(4) \[ 12 - 9 + \frac{27}{4} - \ldots \]

(5) \[ \sum_{j=1}^{\infty} \frac{3}{5^j} \]

(6) \[ \sum_{j=0}^{\infty} 4 \left( \frac{2}{3} \right)^j \]