Back to $f(x) = \cos x$ ...
In class, we found a polynomial that can be extended to give

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Let $P_n(x)$ refer to the polynomial approximating the function f which has "order of contact n" at a point $x = a$. That is to say, $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, 3, \ldots, n$, and here we have $a = 0$.

For example, with $f(x) = \cos x$, we have

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{and} \quad P_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

Note that "n" can cause problems. Does it mean the nth term? the order of contact? the value of n in the summation? Be careful!

(1) If we choose $P_4(x)$, giving us 3 non-zero terms, evaluate $f(x)$ and $P_4(x)$ for the following values of $x = a$.

	$f(a)$	$g(a) = P_4(a)$
$a = 2$		
$a = 1$		
$a = 0.5$		

For roughly what values of x does P_4 seem to be a good approximation of f?

(2) Again using 3 non-zero terms, let $x = 1$. We want to find an upper bound for the error.

(a) First, find the actual error (according to your calculator) by evaluating $|\cos 1 - P_4(1)|$.

(b) Use the alternating series error approximation to find an upper bound for the error.

(3) Still using 3 terms of P, what are the possibilities for x if the error is to be less than 0.00005?

Set the window on the calculator so that $-7 \le x \le 7$ and $-2 \le y \le 2$.

(4) Sketch the graphs of $P_4(x)$ and $f(x)$ below.

For what values of x does P_4 seem to be a fairly good approximation of f ?

(5) Sketch the graphs of $P_8(x)$ and $f(x)$ below.

For what values of x does P_8 seem to be a fairly good approximation of f ?

(6) Sketch the graphs of $P_{12}(x)$ and $f(x)$ below.

For what values of x does P_{12} seem to be a fairly good approximation of f ?

(7) Based on these graphs, can you make any guesses about the values of x for which the infinite series P converges?

Do you think that there is a value of n such that $P_n(x)$ will have an error less than 0.0001 for all x in the interval $-1000 \le x \le 1000$? Why or why not?

(8) Determine the values of x for which the infinite series converges by using an appropriate test.

(9) Compare this result for $y = \cos(x)$ to the intervals found for $y = \ln(1 + x)$ and $y = \tan^{-1}x$.

Can you think of any possible explanations for this distinction?

More on error analysis...

(10) Using $P_{14}(x)$ and $x = 6$, find an upper bound for the error.

(11) Again using $P_{14}(x)$, what are the possibilities for x if the error is to be less than 0.0005?

(12) If $x = 4$, how many terms are necessary to be sure that the error is less than .01?