The Entangling Properties of Knots and Links

Comparing Quantum Entanglement and Topological Entanglement

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1. **Major Results**
   - Knot Invariants

2. **Definitions and Concepts**
   - Knots and Links
   - Knot Polynomials
   - Entanglement

3. **The Hypothesis**
   - Q.L.I’s

4. **Theorem and Proof**
   - Theorem
Knot Invariants Need Quantum Entanglement

**Theorem**

*Non-entangling $R$-matrices cannot form topological invariants*

- Entanglement: acting on a particle at one place will influence it very, very far away.

- Non-entangling: a physical process (operator) that cannot form entangled states from non-entangled states

- Knot Invariants: A unique property of a knot.

- This entire investigation is motivated by a pun!
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Knots and Links

- Knots and links $\iff$ rubber sheet geometry (topology)
- A knot acts just like a closed loop of rope.

**Figure:** Examples of Knots and Links
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Using Polynomials to Distinguish Knots

- Each knot has a unique Jones polynomial.
- The Jones polynomial has many connections to quantum physics.
- The Jones polynomial is related to the Kauffman bracket polynomial - the knot invariant we used in this study.

\[
\begin{align*}
\includegraphics[width=\textwidth]{bracket_relation.png}
\end{align*}
\]

**Figure:** The Bracket Relation
Reidemeister Moves and Topology

- Satisfying these moves will preserve topological properties

Figure: The Reidemeister moves
The Bracket Polynomial

- The bracket satisfies Reidemeister moves II and III.
- It does NOT satisfy Reidemeister I.
- We can introduce a corrective factor to account for Reidemeister I.

\[ X(\includegraphics{diagram1}) = (-A^3)^{-w(\includegraphics{diagram2})} \cdot \langle \includegraphics{diagram3} \rangle \]

\[ = (-A^3)^{-w(\includegraphics{diagram4})-1} \cdot (-A^{-3} \langle \includegraphics{diagram5} \rangle) \]

\[ = (-A^3)^{-w(\includegraphics{diagram6})} \cdot \langle \includegraphics{diagram7} \rangle \]

\[ = X(\includegraphics{diagram8}) \]
Example Calculation of the Bracket

\[ \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle = A \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle \]

\[ = A^2 \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle + 2 \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle + A^{-2} \left\langle \begin{array}{c} \includegraphics[width=1cm]{hopf_link} \end{array} \right\rangle \]

\[ = (A^2 + A^{-2}) \cdot (-A^2 - A^{-2}) + 2 \]

\[ = -A^4 - A^{-4} \]

**Figure:** Bracket of the Hopf link
The general form of the bracket is given by

\[ \langle K \rangle = \sum_S \langle K|S \rangle d^{|S|-1} \]

where \( d = (-A^2 - A^{-2}) \).

The normalization of the oriented bracket is given by

\[ f_k = (-A^3)^{-w(K)} \langle K \rangle \in \mathbb{Z}[A, A^{-1}] \]

where \( w(K) \) is the writhe (sum of oriented crossings).
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Quantum Mechanics is a probabilistic theory of nature.

Quantum processes are transformations on a complex vector space.

Figure: Quantum Mechanics
Entanglement

- Entanglement is the correlation of quantum states.
- Mathematically, we say that a quantum process $G$ is entangling if there is a vector

$$|\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \in V \otimes V$$

such that $G|\alpha\beta\rangle$ cannot be written as a tensor product.

Remark

We have methods to measure how entangling an operator is.
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Quantum Link Invariants

- Knots can be represented by particles moving through space with respect to time.
- These particles are created in pairs (cups) and annihilate subsequently in pairs (caps).

**Figure:** A knot formed from vacuum-vacuum processes
Diagrammatically, we can break up a knot into pieces.

Each piece can be given a "quantum" matrix representation.

\[ R = \text{Overcrossing} \]  \hspace{1cm} (1)

\[ \bar{R} = \text{Undercrossing} \]  \hspace{1cm} (2)

\[ M = \text{Cup or Cap} \]  \hspace{1cm} (3)

**Figure**: Parts of a knot
**The Entangling Properties of Knots and Links**

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**Summary**

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**Figure:** Quantum link invariants in action!
Cups/caps $M$ and crossings $R$ in knots can be given matrix representations.

\[
R = \begin{pmatrix}
    (A^{-1}) & 0 & 0 & 0 \\
    0 & (-A^2 + A^{-1}) & A & 0 \\
    0 & A & 0 & 0 \\
    0 & 0 & 0 & (A^{-1})
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
    0 & (iA) \\
    (-iA^{-1}) & 0
\end{pmatrix}
\]
Statement of the Problem

- From an $R$ matrix like the one shown, topological invariants can be produced.
- All the examples we can calculate show that entanglement is a necessary condition.

Conjecture
Can topology and entanglement be linked through this process?
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Major Results

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Non-entangling R matrices cannot be used to produce topological invariants of quantum links
Proof (Special Case)
Quantum link invariants need entanglement in order to be topological invariants.

Topology, Knot Theory, and Quantum Theory are intimately related.

We still do not understand the extent of the relationship between entanglement and topology.

**Outlook**

- We may be able to employ some techniques from gauge theory and algebraic topology to better understand entanglement strength.
Further Reading

