Mathematical Games, Puzzles and Diversions

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The Game of Fifty

Players take turns choosing numbers 1, 2, 3, 4, 5, or 6 - keeping a running total (the same number can be chosen repeatedly). The one who gets to 50 wins.

Does player one or player two have an advantage? Is there a strategy to always win?
NOTE THAT I CAN FORCE THE SUM TO GO UP 7 AFTER EACH TURN – IF MY OPPONENT PICKS NUMBER $n$, I PICK $7 - n$. THEREFORE, THE WINNING STRATEGY IS TO GO FIRST, CHOOSING 1, THEN TAKE $7 - n$ EACH TIME MY OPPONENT TAKES $n$. 
The Game of 27,371

Players take turns choosing numbers 1, 2, 3, ...9998, or 9999 keeping a running total (the same number can be chosen repeatedly). The one who gets to 27,371 wins.
Topological Puzzle #1
Tie a knot.

To make it more challenging, grab the ends first, and do not let go.
Play a few games:

“Cribbage without cards”

Two player game. Players take turns selecting numbers from 1, 2, 3, ..., 9, without replacement. The first player to get exactly 3 numbers that add to 15 wins the game.
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Reverse Tic-Tac-Toe:
Three in a row loses
WHO ATTENDED ED BURGER’S MMC TALK IN OCTOBER 2013?
Two Player Game

Start with any number of coins and alternate taking coins following the rules below. The player who takes the last coin wins.

Rules:
1. The first player may take any amount of coins, provided they leave at least one coin.
2. Thereafter, a player may take up to twice as many as their opponent took on that turn.
Zeckendorf's Theorem:
Every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers.
HOW HIGH CAN YOU GO?
START WITH AN INFINITE GRID. PLACE AS MANY PIECES AS YOU PLEASE ANYWHERE BELOW THE “STARTING” LINE. JUMP HORIZONTALLY OR VERTICALLY AS IN CHECKERS (NO DIAGONAL JUMPS), REMOVING THE PIECE THAT WAS JUMPED. THE OBJECT IS TO GET A PIECE AS HIGH AS POSSIBLE.
Cover the grid as shown
We choose $0 < x < 1$ as follows:

1. $x^n + x^{n-1} \geq x^{n-2} \Rightarrow x^2 + x \geq 1$
   
   (e.g. jumping up or to right from left of center)

2. $x^n + x^{n-1} \rightarrow x^n \Rightarrow x + 1 \geq x$
   
   (e.g. jumping right over the center)

3. $x^{n-1} + x^n \geq x^{n+1} \Rightarrow x + 1 \geq x^2$
   
   (e.g. jumping right to the right of center)

Note: 2. and 3. are trivially true.

1. Is true when $x = \frac{1}{\phi} = \frac{-1 + \sqrt{5}}{2}$
So when $x = \frac{1}{\phi}$, no jump raises the sum of pieces left on the board.
So the sum of squares that are covered by pieces never increases.

What is the maximum sum that we can start with?
Suppose all squares below the line are used:

We start with a sum of at most:

\[
x^6 + 3x^7 + 5x^8 + 7x^9 + L \ L
= [1 + x + x^2 + L] \cdot [x^6 + 2x^7 + 2x^8 + 2x^9 + L]
= [1 + x + x^2 + L] \cdot [x^6 + 2x^7 \cdot (1 + x + x^2 + L)]
= [\frac{1}{1-x}] \cdot [x^6 + 2x^7 \cdot (\frac{1}{1-x})]
= 1 \text{ when } x = \frac{1}{\phi}
\]

So, we can never get past the sixth row!!
What happens if diagonal jumps are allowed?
Topological Puzzle #2

All tied up

Now, how did she do that?
Topological Puzzle #3