The Box Problem
an introduction

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Graphic by Roger Kirchner, “Tin Box with Maximum Volume”
http://demonstrations.wolfram.com/TinBoxwithMaximumVolume/
Wolfram Demonstrations Project
The Box Problem

Perhaps the famous Box Problem is familiar, or perhaps not. Just in case, here’s the basic idea.

Start with a rectangular piece of paper. A square is a possibility. Cut out small squares from the four corners. Fold up the sides to create an box with no top. What is the size of the squares that should be removed in order to maximize the volume of the box?

There are many possible extensions to make the problem more interesting and more challenging. One may start with squares or with rectangles. It is possible to generalize the problem with different parameters. It is also possible to start with shapes other than rectangles and remove corresponding shapes from the vertices of the initial shape. This problem is frequently used in calculus classes when beginning the study of optimization.

Here, we’ll take a very gentle introduction that can be used in early algebra classes. This will allow students to build boxes, work with data, finite differences, and regression.

Note: Scissors and some tape will be needed.

Check out:

Roger B. Kirchner
"Tin Box with Maximum Volume"
http://demonstrations.wolfram.com/TinBoxWithMaximumVolume/
Wolfram Demonstrations Project
Published: July 29, 2009

This will need a free player program – also useful for many other demos!
to make the box
(1) You have been given a piece of paper with a 19 x 19 grid on it. Cut out the grid. Each student should be given an integer from 1 to 9.

With your integer \(a\), cut a square from each corner of the grid that measures \(a\) units on a side.

For example, if your integer is 3, then cut a 3x3 square from each of the four corners. That is, cut the shaded corners below.

[Diagram of a 19x19 grid with shaded corners cut out]

Fold the sides up. In the 3x3 example above, fold along the dashed segments. Use tape to hold your box together.

For your box, find the perimeter of the base, the area of the base, and the volume of your box.

Perimeter of the base: _________________

Area of the base: _________________

Volume of the box: _________________

Look around the room at the other boxes created by other students.

(2) When all students have measurements for their boxes, fill in the chart on the next page with the appropriate numbers. (Leave the other columns with \(\Delta\)’s blank right now.)
(3) After you have all the numbers filled in for height, perimeter, area, and volume, it’s time to make graphs using the graph on the next page.

Let $h$ be the height of the box represent the independent variable on the horizontal axis. Make three separate graphs – for the perimeter of the base, the area of the base, and the volume of the box – on the same set of axes.

Note that the units on the y-axis are different for each graph. Assume that each box on the grid represents a foot, the units will be as follows.

For the perimeter of the base, the units are ________________.

For the area of the base, the units are ________________.

For the volume of the box, the units are ________________.

<table>
<thead>
<tr>
<th>Height of the box</th>
<th>Perimeter of the base</th>
<th>$\Delta P$</th>
<th>Area of the base</th>
<th>$\Delta A$</th>
<th>$\Delta^2 A$</th>
<th>Volume of the box</th>
<th>$\Delta V$</th>
<th>$\Delta^2 V$</th>
<th>$\Delta^3 V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Now return to your chart. It’s time to fill in the remaining numbers in the chart.

a. In the column labeled $\Delta P$ (which is called the *first order difference*), calculate the *changes* in the perimeter. For example, in the second row for the column labeled $\Delta P$, you should enter the value of (Perimeter of the base for height 1) – (Perimeter of the base for height 0).

b. In the column labeled $\Delta A$ (another first order difference), calculate the *changes* in the area. For example, in the second row for the column labeled $\Delta A$, you should enter the value of (Area of the base for height 1) – (Area of the base for height 0).

In the column labeled $\Delta^2 A$ (which is called the *second order difference*), calculate the *changes* $\Delta A$. For example, in the third row for the column labeled $\Delta^2 A$, you should enter the value of ($\Delta A$ from the third row) – ($\Delta A$ from the second row).

c. Compute the values for $\Delta V$ (the first order difference), $\Delta^2 V$ (the second order difference), and $\Delta^3 V$ (the *third order difference*) in a similar manner to what you did for $\Delta A$ and $\Delta^2 A$.

Next, it’s time to write formulas for each of functions we’re examining. Think carefully about how each function was created.

a. The perimeter, $P$, of the base of the box as a function of the height, $x$, of the box.

$$P(x) = \text{______________________________}$$

b. The area, $A$, of the base of the box as a function of the height, $x$, of the box.

$$A(x) = \text{______________________________}$$

c. The volume, $V$, of the box as a function of the height, $x$, of the box.

$$V(x) = \text{______________________________}$$
(6) How does the table with finite differences connect to your formulas for perimeter, area, and volume?

(7) Use your calculator to find appropriate regression equations.
   a. The perimeter, $P$, of the base of the box as a function of the height, $x$, of the box.
      
      \[ P(x) = \ \text{________} \]
   b. The area, $A$, of the base of the box as a function of the height, $x$, of the box.
      
      \[ A(x) = \ \text{________} \]
   c. The volume, $V$, of the box as a function of the height, $x$, of the box.
      
      \[ V(x) = \ \text{________} \]

(8) Do the regression equations from your calculator agree with the formulas that you created? Check the graphs of these equations to be sure that these formulas agree with the data in your chart.